



OPERATIONS ON P-ORDER AND R-ORDER OF INTERNAL CUBIC SOFT MATRICES

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Abstract:

In this paper, we introduce the notion of P-order (R-order) of internal cubic soft matrices. Further, we develop union and intersection of P-order(R-order) of internal cubic soft matrices and their related properties have been investigated.

Key Words: Cubic Soft Sets, Cubic Soft Matrices, P-Order Internal Cubic Soft Matrices & R-Order Internal Cubic Soft Matrices.

1. Introduction:

Fuzzy set theory was proposed by Lotifi A. Zadeh [12] and it has extensive applications in various fields. In 1999, Molodstov[8] introduced the novel concept of soft sets and established the fundamental results of the new theory. In 2003, Maji *et al.*[6] studied some properties of soft sets. Pei and Miao [10] and Chen [1]*et al.*, improved the work of Maji *et al.* [5, 6]. In [3], Jun *et al.*, introduced a new notion of cubic set which is a combination of fuzzy set and interval-valued fuzzy set and also investigated several properties of cubic sets.

Muhiuddin and Al-roqi [9], introduced the concepts of internal, external cubic soft sets, P-cubic (R-cubic) soft subsets, R-union(R-intersection, P-union and P-intersection) of cubic soft sets and the complement of a cubic soft sets. They also investigated several related properties and applied the notion of cubic soft sets to BCK/BCI-algebras.

Fuzzy matrix was introduced by Thomason [11] and the concept of uncertainty was discussed by using fuzzy matrices. Different concepts and ideas of fuzzy matrices have been given earlier mainly by Kim, Meenakshi and Thomson [4, 7, 11]. Fuzzy matrix plays a vital role in fuzzy modeling, fuzzy diagnosis and fuzzy controls. It also has applications in fields like psychology, medicine, economics and sociology.

In the early investigation, Chinnadurai and Barkavi introduced a new concepts of cubic soft matrix, internal cubic soft matrix, external cubic soft matrix and discussed various result in[2].

In this paper, the new definition of P-order internal cubic soft matrices and R-order internal cubic soft matrices have been introduced, along with some algebraic properties are discussed with suitable illustrations.

2. Preliminaries:

In this section we recall some basic definitions and results which will be needed in the sequel.

Definition 2.1 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set over U is defined to be a pair (Φ, A) where Φ is a mapping from A to $P(U)$ and $A \subseteq E$. Then the pair (Φ, A) can be represented as, $(\Phi, A) = \{\Phi(e)/e \in A\}$ where $\Phi(e) = \left\{ \langle u, \tilde{A}_e(u), \lambda_e(u) \rangle / u \in U, e \in A \right\}$ is a cubic soft set in which $\tilde{A}_e(u)$ is the interval valued fuzzy set and $\lambda_e(u)$ is a fuzzy set.

Definition 2.2 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set (Φ, A) over U is said to be an internal cubic soft set if $A_e^-(u) \leq \lambda_e(u) \leq A_e^+(u)$ for all $e \in A$ and for all $u \in U$.

Definition 2.3 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set (Φ, A) over U is said to be an external cubic soft set if $\lambda_e(u) \notin (A_e^-(u), A_e^+(u))$ for all $e \in A$ and for all $u \in U$.

Definition 2.4 [9] Let U be an initial universal set and E be a set of parameters. For any subsets A and B of E , (Φ, A) and (Γ, B) be cubic soft sets over U .

1. The R-union of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cup B$ and

$$He) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cup_R \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$ and it is denoted by $(H, C) = (\Phi, A) \cup_R (\Gamma, B)$.

2. The R-intersection of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cap B$ and

$$He) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cap_R \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$ and it is denoted by $(H, C) = (\Phi, A) \cap_R (\Gamma, B)$.

Definition 2.5 [9] Let U be an initial universal set and E be a set of parameters. For any subsets A and B of E , (Φ, A) and (Γ, B) be cubic soft sets over U .

1. The P-union of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cup B$ and

$$He) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cup_P \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(H, C) = (\Phi, A) \cup_P (\Gamma, B)$.

2. The P-intersection of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cap B$ and

$$He) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cap_P \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(H, C) = (\Phi, A) \cap_P (\Gamma, B)$.

Definition 2.6 [9] Let U be an initial universal set and E be a set of parameters. The complement of a cubic soft set (Φ, A) over U is denoted by $(\Phi, A)^c$ and defined by

$$(\Phi, A)^c = (\Phi^c, -A) \text{ where } \Phi^c : -A \rightarrow \mathcal{XP}(U) \text{ and } (\Phi, A)^c = \{\Phi^c(e)/e \in A\} \text{ where } \Phi^c(e) = \langle u, \tilde{A}_e^c(u), \lambda_e^c(u) \rangle / u \in U, e \in A\}$$

Definition 2.7 [7] A matrix $A = [a_{ij}]_{m \times n}$ is said to be fuzzy matrix if $a_{ij} \in [0, 1], 1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 2.8 [7] For any two fuzzy matrices $A = [a_{ij}], B = [b_{ij}]$ and a scalar $k \in F$. Then,

- (i) $A + B = [\sup\{a_{ij}, b_{ij}\}] = \vee[a_{ij}, b_{ij}]$.
- (ii) $AB = [\sup\{\inf\{a_{ij}, b_{ij}\}\}] = \vee\{\wedge[a_{ij}, b_{ij}]\}$.
- (iii) $kA = [\inf\{k, a_{ij}\}] = \wedge[k, a_{ij}]$.

Definition 2.9 [2] Let $U = \{u_1, u_2, \dots, u_m\}$ be an initial universal set and $E = \{e_1, e_2, \dots, e_n\}$ be a set of parameters. Let $A \subseteq E$. Then cubic soft set (Φ, A) can be expressed in matrix form as

$$A^c = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

such that $A^c = [a_{ij}] = \langle \tilde{A}_{e_j}^c(u_i), \lambda_{e_j}^c(u_i) \rangle = \langle \tilde{A}_{ij}^c, \lambda_{ij}^c \rangle$ which is called an $m \times n$ cubic soft matrix (shortly CS-matrix or CSM) of the cubic soft set (Φ, A) .

According to this definition, a cubic soft set (Φ, A) is uniquely characterized by matrix $[a_{ij}]_{m \times n}$ where

$i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

Example 2.10 Let $U = \{u_1, u_2, u_3, u_4\}$ is a set of cars and $A = \{e_1, e_2, e_3\}$ is a set of parameters, which stands for mileage, engine and prize respectively. Then cubic soft set is defined as

$$(\Phi, A) = \left\{ \begin{array}{l} [e_1, (u_1, \langle [0.5, 0.8]0.6 \rangle), (u_2, \langle [0.1, 0.7]0.5 \rangle), (u_3, \langle [0.2, 0.6]0.9 \rangle), (u_4, \langle [0.3, 0.9]0.4 \rangle)], \\ [e_2, (u_1, \langle [0.2, 0.5]0.3 \rangle), (u_2, \langle [0.3, 0.6]0.7 \rangle), (u_3, \langle [0.2, 0.7]0.2 \rangle), (u_4, \langle [0.3, 0.5]0.1 \rangle)], \\ [e_3, (u_1, \langle [0.1, 0.8]0.4 \rangle), (u_2, \langle [0.6, 0.7]0.9 \rangle), (u_3, \langle [0.2, 0.9]0.5 \rangle), (u_4, \langle [0.3, 0.7]0.4 \rangle)] \end{array} \right\}.$$

Then the CS-matrix A^c is written as,

$$A^c = \begin{bmatrix} \langle [0.5, 0.8]0.6 \rangle & \langle [0.2, 0.5]0.3 \rangle & \langle [0.1, 0.8]0.4 \rangle \\ \langle [0.1, 0.7]0.5 \rangle & \langle [0.3, 0.6]0.7 \rangle & \langle [0.6, 0.7]0.9 \rangle \\ \langle [0.2, 0.6]0.7 \rangle & \langle [0.2, 0.7]0.2 \rangle & \langle [0.2, 0.9]0.5 \rangle \\ \langle [0.3, 0.9]0.4 \rangle & \langle [0.3, 0.5]0.1 \rangle & \langle [0.3, 0.7]0.4 \rangle \end{bmatrix}.$$

Definition 2.11 [2] Let $A^c = [a_{ij}] \in CSM_{m \times n}$. Then A^c is an internal cubic soft matrix (ICSM), if $\tilde{A}_{ij}^{a-} \leq \lambda_{ij}^a \leq \tilde{A}_{ij}^{a+}$ for all i, j .

Definition 2.12 [2] Let $A^c = [a_{ij}]_{m \times n}$, $B^c = [b_{ij}]_{m \times n}$ be the two cubic soft matrix of order $m \times n$. Then P-order matrix is denoted and defined as $[a_{ij}] \subseteq_P [b_{ij}]$, if $\tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

Definition 2.13 [2] Let $A^c = [a_{ij}]_{m \times n}$, $B^c = [b_{ij}]_{m \times n}$ be the two cubic soft matrix of order $m \times n$. Then R-order matrix is denoted and defined as $[a_{ij}] \subseteq_R [b_{ij}]$, if $\tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

Definition 2.14 [2] Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle$, $B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \in CSM_{m \times n}$. Then

1. P-union of A^c and B^c is denoted by $A^c \vee_P B^c$ and defined as $A^c \vee_P B^c = C^c$, if $C^c = [c_{ij}] = \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$, where $\tilde{C}_{ij}^c = \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j .
2. P-intersection of A^c and B^c is denoted by $A^c \wedge_P B^c$ and defined as $A^c \wedge_P B^c = C^c$, if $C^c = [c_{ij}] = \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$, where $\tilde{C}_{ij}^c = \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j .
3. R-union of A^c and B^c is denoted by $A^c \vee_R B^c$ and defined as $A^c \vee_R B^c = C^c$, if $C^c = [c_{ij}] = \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$, where $\tilde{C}_{ij}^c = \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j .
4. R-intersection of A^c and B^c is denoted by $A^c \wedge_R B^c$ and defined as $A^c \wedge_R B^c = C^c$, if $C^c = [c_{ij}] = \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$, where $\tilde{C}_{ij}^c = \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j .

3. P-order and R-order of Internal Cubic Soft Matrices:

In this section we define P-order and R-order of internal cubic soft matrix and discussed some algebraic properties.

Definition 3.1 Let $A^c = \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle$ and $B^c = \langle \tilde{B}_{ij}^b, \mu_{ij}^b \rangle$ be internal cubic soft matrices, if $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j . Then it is defined as P-order internal cubic soft matrices and it is denoted by $ICSM^P$.

Example 3.2 Let $U = \{u_1, u_2\}$ be the universal set and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. Let $A, B \subseteq E$. Then

$$(\Phi, A) = \left\{ \begin{array}{l} [e_1, (u_1, \langle [0.2, 0.8]0.5 \rangle), (u_2, \langle [0.3, 0.7]0.4 \rangle)], \\ [e_2, (u_1, \langle [0.4, 0.9]0.6 \rangle), (u_2, \langle [0.3, 0.8]0.35 \rangle)] \end{array} \right\}$$

and

$$(\Phi, B) = \left\{ \begin{array}{l} [e_1, (u_1, \langle [0.4, 0.9], 0.7 \rangle), (u_2, \langle [0.5, 0.8], 0.6 \rangle)] \\ [e_2, (u_1, \langle [0.6, 1], 0.8 \rangle), (u_2, \langle [0.7, 0.9], 0.75 \rangle)] \end{array} \right\}.$$

Then P-order internal cubic soft matrices is defined as follows,

$$A^\zeta = \begin{bmatrix} \langle [0.2, 0.8], 0.5 \rangle & \langle [0.4, 0.9], 0.6 \rangle \\ \langle [0.3, 0.7], 0.4 \rangle & \langle [0.3, 0.8], 0.5 \rangle \end{bmatrix} \subseteq_P B^\zeta = \begin{bmatrix} \langle [0.4, 0.9], 0.7 \rangle & \langle [0.6, 1], 0.8 \rangle \\ \langle [0.5, 0.8], 0.6 \rangle & \langle [0.7, 0.9], 0.75 \rangle \end{bmatrix}.$$

(i.e) $A^\zeta \subseteq_P B^\zeta$.

Definition 3.3 Let $A^\zeta = \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle$ and $B^\zeta = \langle \tilde{B}_{ij}^b, \mu_{ij}^b \rangle$ be internal cubic soft matrices, if

$A^\zeta \subseteq_R B^\zeta \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j. Then it is defined as R-order internal cubic soft matrices and it is denoted by $ICSM^R$.

Example 3.4 Let $U = \{u_1, u_2\}$ be the universal set and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. Let $A, B \subseteq E$. Then

$$(\Phi, A) = \left\{ \begin{array}{l} [e_1, (u_1, \langle [0.2, 0.4], 0.3 \rangle), (u_2, \langle [0.1, 0.6], 0.5 \rangle)] \\ [e_2, (u_1, \langle [0.4, 0.8], 0.7 \rangle), (u_2, \langle [0.2, 0.4], 0.4 \rangle)] \end{array} \right\}$$

and

$$(\Phi, B) = \left\{ \begin{array}{l} [e_1, (u_1, \langle [0.3, 0.7], 0.3 \rangle), (u_2, \langle [0.3, 0.9], 0.4 \rangle)] \\ [e_2, (u_1, \langle [0.5, 1], 0.6 \rangle), (u_2, \langle [0.3, 0.8], 0.35 \rangle)] \end{array} \right\}.$$

Then R-order internal cubic soft matrices is defined as follows,

$$A^\zeta = \begin{bmatrix} \langle [0.2, 0.4], 0.3 \rangle & \langle [0.4, 0.8], 0.7 \rangle \\ \langle [0.1, 0.6], 0.5 \rangle & \langle [0.2, 0.4], 0.4 \rangle \end{bmatrix} \subseteq_R B^\zeta = \begin{bmatrix} \langle [0.3, 0.7], 0.3 \rangle & \langle [0.5, 1], 0.6 \rangle \\ \langle [0.3, 0.9], 0.4 \rangle & \langle [0.3, 0.8], 0.35 \rangle \end{bmatrix}.$$

(i.e) $A^\zeta \subseteq_R B^\zeta$.

Definition 3.5 Let $A^\zeta, B^\zeta \in ICSM_{m \times n}^P$. Then

a) Union of A^ζ, B^ζ is defined by $A^\zeta \vee_P B^\zeta = C^\zeta = [c_{ij}]$, where

$$C^\zeta = \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all } i, j.$$

b) Intersection of A^ζ, B^ζ is defined by $A^\zeta \wedge_P B^\zeta = C^\zeta = [c_{ij}]$, where

$$C^\zeta = \left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all } i, j.$$

Example 3.6 Consider Example 3.2. Then union and intersection of A^ζ, B^ζ are given by,

a) $A^\zeta \vee_P B^\zeta = \begin{bmatrix} \langle [0.4, 0.9], 0.7 \rangle & \langle [0.6, 1], 0.8 \rangle \\ \langle [0.5, 0.8], 0.6 \rangle & \langle [0.7, 0.9], 0.75 \rangle \end{bmatrix}.$

b) $A^\zeta \wedge_P B^\zeta = \begin{bmatrix} \langle [0.2, 0.8], 0.5 \rangle & \langle [0.4, 0.9], 0.6 \rangle \\ \langle [0.3, 0.7], 0.4 \rangle & \langle [0.3, 0.8], 0.35 \rangle \end{bmatrix}.$

Definition 3.7 Let $A^\zeta, B^\zeta \in ICSM_{m \times n}^R$. Then

a) Union of A^ζ, B^ζ is defined by $A^\zeta \vee_R B^\zeta = C^\zeta = [c_{ij}]$, where

$$C^\zeta = \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all } i, j.$$

b) Intersection of A^ζ, B^ζ is defined by $A^\zeta \wedge_R B^\zeta = C^\zeta = [c_{ij}]$, where

$$C^c = \left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all } i, j.$$

Example 3.8 Consider Example 3.4. Then union and intersection of A^c, B^c are given by,

$$a) A^c \vee_R B^c = \begin{bmatrix} \langle [0.3, 0.7], 0.3 \rangle & \langle [0.5, 1], 0.6 \rangle \\ \langle [0.3, 0.9], 0.4 \rangle & \langle [0.3, 0.8], 0.35 \rangle \end{bmatrix}.$$

$$b) A^c \wedge_R B^c = \begin{bmatrix} \langle [0.2, 0.4], 0.3 \rangle & \langle [0.4, 0.8], 0.7 \rangle \\ \langle [0.1, 0.6], 0.5 \rangle & \langle [0.2, 0.4], 0.4 \rangle \end{bmatrix}.$$

Proposition 3.9 Let $A^c, B^c, C^c \in ICSM_{m \times n}^P$. Then

(i) $A^c \vee_P A^c = A^c$.

(ii) $A^c \vee_P B^c = B^c$.

(iii) $(A^c \vee_P B^c) \vee_P C^c = A^c \vee_P (B^c \vee_P C^c) = C^c$.

Proof. Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle, B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle, C^c = \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \in ICSM_{m \times n}^P$.

Then (i) $A^c \vee_P A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_P \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle$ for all i, j .

Since $A^c \subseteq_P A^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{A}_{ij}^a$ and $\lambda_{ij}^a \leq \lambda_{ij}^a$ for all i, j .

By Definition 3.5(a),

(ii) $A^c \vee_P B^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_P \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle$ for all i, j .

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} A^c \vee_P B^c &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ &= B^c. \end{aligned}$$

(iii) $(A^c \vee_P B^c) \vee_P C^c = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_P \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right) \vee_P C^c$.

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} (A^c \vee_P B^c) \vee_P C^c &= \left(\left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right) \vee_P C^c \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \vee_P C^c \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle. \end{aligned}$$

Since $B^c \subseteq_P C^c \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \leq \gamma_{ij}^c$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} (A^c \vee_P B^c) \vee_P C^c &= \left\langle [\max\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= C^c. \end{aligned}$$

Similarly, $A^c \vee_P (B^c \vee_P C^c) = C^c$.

Hence, $(A^c \vee_P B^c) \vee_P C^c = A^c \vee_P (B^c \vee_P C^c) = C^c$.

Proposition 3.10 Let $A^c, B^c, C^c \in ICSM_{m \times n}^R$. Then

(i) $A^c \vee_R A^c = A^c$.

(ii) $A^c \vee_R B^c = B^c$.

(iii) $(A^c \vee_R B^c) \vee_R C^c = A^c \vee_R (B^c \vee_R C^c) = C^c$.

Proof. Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle, B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle, C^c = \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \in ICSM_{m \times n}^R$.

Then (i) $A^c \vee_R A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_R \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle$ for all i, j .

Since $A^c \subseteq_R A^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{A}_{ij}^a$ and $\lambda_{ij}^a \geq \lambda_{ij}^a$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} A^c \vee_R A^c &= \langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{A}_{ij}^{a+}\}], \min\{\lambda_{ij}^a, \lambda_{ij}^a\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

(ii) $A^c \vee_R B^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle$ for all i, j .

Since $A^c \subseteq_R B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} A^c \vee_R B^c &= \langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \\ &= \langle \tilde{B}_{ij}^b, \mu_{ij}^b \rangle \\ &= B^c. \end{aligned}$$

(iii) $(A^c \vee_R B^c) \vee_R C^c = \langle \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \rangle \vee_R C^c$.

Since $A^c \subseteq_R B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} (A^c \vee_R B^c) \vee_R C^c &= \langle \langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \rangle \vee_R C^c \\ &= \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \vee_R C^c \\ &= \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \vee_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle. \end{aligned}$$

Since $B^c \subseteq_R C^c \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} (A^c \vee_R B^c) \vee_R C^c &= \langle [\max\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \min\{\mu_{ij}^b, \gamma_{ij}^c\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \\ &= \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle \\ &= C^c. \end{aligned}$$

Similarly, $A^c \vee_R (B^c \vee_R C^c) = C^c$.

Hence, $(A^c \vee_R B^c) \vee_R C^c = A^c \vee_R (B^c \vee_R C^c) = C^c$.

Proposition 3.11 Let $A^c, B^c, C^c \in ICSM_{m \times n}^P$. Then

(i) $A^c \wedge_P A^c = A^c$.

(ii) $A^c \wedge_P B^c = A^c$.

(iii) $(A^c \wedge_P B^c) \wedge_P C^c = A^c \wedge_P (B^c \wedge_P C^c) = A^c$.

Proof. Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle, B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle, C^c = \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \in ICSM_{m \times n}^P$.

Then (i) $A^c \wedge_P A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle$ for all i, j .

Since $A^c \subseteq_P A^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{A}_{ij}^a$ and $\lambda_{ij}^a \leq \lambda_{ij}^a$ for all i, j .

By Definition 3.5(b),

$$\begin{aligned} A^c \wedge_P A^c &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{A}_{ij}^{a+}\}], \min\{\lambda_{ij}^a, \lambda_{ij}^a\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

(ii) $A^c \wedge_P B^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle$ for all i, j .

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

By Definition 3.5(b),

$$\begin{aligned} A^c \wedge_P B^c &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

(iii) $(A^c \wedge_P B^c) \wedge_P C^c = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right) \wedge_P C^c$.

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

By Definition 3.5(b),

$$\begin{aligned} (A^c \wedge_P B^c) \wedge_P C^c &= \left(\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \right) \wedge_P C^c \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P C^c \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle. \end{aligned}$$

Since $A^c \subseteq_P C^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j .

Again by Definition 3.5(b),

$$\begin{aligned} (A^c \wedge_P B^c) \wedge_P C^c &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

Similarly, $A^c \wedge_p (B^c \wedge_p C^c) = A^c$.

Hence, $(A^c \wedge_p B^c) \wedge_p C^c = A^c \wedge_p (B^c \wedge_p C^c) = A^c$.

Proposition 3.12 Let $A^c, B^c, C^c \in ICSM_{m \times n}^R$. Then

(i) $A^c \wedge_R A^c = A^c$.

(ii) $A^c \wedge_R B^c = A^c$.

(iii) $(A^c \wedge_R B^c) \wedge_R C^c = A^c \wedge_R (B^c \wedge_R C^c) = A^c$.

Proof. Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle, B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle, C^c = \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \in ICSM_{m \times n}^R$.

Then (i) $A^c \wedge_R A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle$ for all i, j .

Since $A^c \subseteq_R A^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{A}_{ij}^a$ and $\lambda_{ij}^a \geq \lambda_{ij}^a$ for all i, j .

By Definition 3.7(b),

$$\begin{aligned} A^c \wedge_R A^c &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{A}_{ij}^{a+}\}], \max\{\lambda_{ij}^a, \lambda_{ij}^a\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

(ii) $A^c \wedge_R B^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle$ for all i, j .

Since $A^c \subseteq_R B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(b),

$$\begin{aligned} A^c \wedge_R B^c &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

(iii) $(A^c \wedge_R B^c) \wedge_R C^c = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right) \wedge_R C^c$.

Since $A^c \subseteq_R B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(b),

$$\begin{aligned} (A^c \wedge_R B^c) \wedge_R C^c &= \left(\langle [\min(\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}), \min(\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+})], \max(\lambda_{ij}^a, \mu_{ij}^b) \rangle \right) \wedge_R C^c \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R C^c \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle. \end{aligned}$$

Since, $A^c \subseteq_R C^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j .

Again by Definition 3.7(b),

$$\begin{aligned} (A^c \wedge_R B^c) \wedge_R C^c &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \rangle \text{ for all } i, j \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

Similarly, $A^{(} \wedge_R (B^{(} \wedge_R C^{(}) = A^{(}$.

Hence, $(A^{(} \wedge_R B^{(}) \wedge_R C^{(} = A^{(} \wedge_R (B^{(} \wedge_R C^{(}) = A^{(}$.

Theorem 3.13 Let $A_1^{(}, A_2^{(}, A_3^{(}, \dots, A_k^{(} \in ICSM_{m \times n}^P$. Since $A_1^{(} \subseteq_P A_2^{(} \subseteq_P A_3^{(} \subseteq_P \dots \subseteq_P A_k^{(}$.

Then $A_1^{(} \vee_P A_2^{(} \vee_P A_3^{(} \vee_P \dots \vee_P A_k^{(} = A_k^{(}$.

Proof. Let

$$A_1^{(} = \langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \rangle, A_2^{(} = \langle [\tilde{A}_{2ij}^{a-}, \tilde{A}_{2ij}^{a+}], \lambda_{2ij}^a \rangle, A_3^{(} = \langle [\tilde{A}_{3ij}^{a-}, \tilde{A}_{3ij}^{a+}], \lambda_{3ij}^a \rangle, \dots,$$

$$A_k^{(} = \langle [\tilde{A}_{kij}^{a-}, \tilde{A}_{kij}^{a+}], \lambda_{kij}^a \rangle \in ICSM_{m \times n}^P.$$

Then, $A_1^{(} \vee_P A_2^{(} \vee_P A_3^{(} \vee_P \dots \vee_P A_k^{(} = (A_1^{(} \vee_P A_2^{(}) \vee_P A_3^{(} \vee_P \dots \vee_P A_k^{(}$.

Since, $A_1^{(} \subseteq_P A_2^{(} \Leftrightarrow \tilde{A}_{1ij}^a \leq \tilde{A}_{2ij}^a$ and $\lambda_{1ij}^a \leq \lambda_{2ij}^a$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} &= \left(\left\langle [\max\{\tilde{A}_{1ij}^{a-}, \tilde{A}_{2ij}^{a-}\}, \max\{\tilde{A}_{1ij}^{a+}, \tilde{A}_{2ij}^{a+}\}], \max\{\lambda_{1ij}^a, \lambda_{2ij}^a\} \right\rangle \right) \vee_P A_3^{(} \vee_P \dots \vee_P A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{2ij}^{a-}, \tilde{A}_{2ij}^{a+}], \lambda_{2ij}^a \right\rangle \right) \vee_P A_3^{(} \vee_P \dots \vee_P A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{2ij}^{a-}, \tilde{A}_{2ij}^{a+}], \lambda_{2ij}^a \right\rangle \vee_P \left\langle [\tilde{A}_{3ij}^{a-}, \tilde{A}_{3ij}^{a+}], \lambda_{3ij}^a \right\rangle \right) \vee_P A_4^{(} \vee_P \dots \vee_P A_k^{(}. \end{aligned}$$

Since, $A_2^{(} \subseteq_P A_3^{(} \Leftrightarrow \tilde{A}_{2ij}^a \leq \tilde{A}_{3ij}^a$ and $\lambda_{2ij}^a \leq \lambda_{3ij}^a$ for all i, j .

Again by Definition 3.5(a),

$$\begin{aligned} &= \left(\left\langle [\max\{\tilde{A}_{2ij}^{a-}, \tilde{A}_{3ij}^{a-}\}, \max\{\tilde{A}_{2ij}^{a+}, \tilde{A}_{3ij}^{a+}\}], \max\{\lambda_{2ij}^a, \lambda_{3ij}^a\} \right\rangle \right) \vee_P A_4^{(} \vee_P \dots \vee_P A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{3ij}^{a-}, \tilde{A}_{3ij}^{a+}], \lambda_{3ij}^a \right\rangle \right) \vee_P A_4^{(} \vee_P \dots \vee_P A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{3ij}^{a-}, \tilde{A}_{3ij}^{a+}], \lambda_{3ij}^a \right\rangle \vee_P \left\langle [\tilde{A}_{4ij}^{a-}, \tilde{A}_{4ij}^{a+}], \lambda_{4ij}^a \right\rangle \right) \vee_P A_5^{(} \vee_P \dots \vee_P A_k^{(}. \end{aligned}$$

Continuing the same process we get,

$$= \left(\left\langle [\tilde{A}_{(k-1)ij}^{a-}, \tilde{A}_{(k-1)ij}^{a+}], \lambda_{(k-1)ij}^a \right\rangle \vee_P \left\langle [\tilde{A}_{kij}^{a-}, \tilde{A}_{kij}^{a+}], \lambda_{kij}^a \right\rangle \right).$$

Since, $A_{(k-1)}^{(} \subseteq_P A_k^{(} \Leftrightarrow \tilde{A}_{(k-1)ij}^a \leq \tilde{A}_{kij}^a$ and $\lambda_{(k-1)ij}^a \leq \lambda_{kij}^a$ for all i, j .

Again by the Definition 3.5(a),

$$\begin{aligned} &= \left(\left\langle [\max\{\tilde{A}_{(k-1)ij}^{a-}, \tilde{A}_{kij}^{a-}\}, \max\{\tilde{A}_{(k-1)ij}^{a+}, \tilde{A}_{kij}^{a+}\}], \max\{\lambda_{(k-1)ij}^a, \lambda_{kij}^a\} \right\rangle \right) \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{kij}^{a-}, \tilde{A}_{kij}^{a+}], \lambda_{kij}^a \right\rangle \right) \\ &= \left(\left\langle \tilde{A}_{kij}^a, \lambda_{kij}^a \right\rangle \right). \\ &= A_k^{(}. \end{aligned}$$

Theorem 3.14 Let $A_1^{(}, A_2^{(}, A_3^{(}, \dots, A_k^{(} \in ICSM_{m \times n}^R$. Since $A_1^{(} \subseteq_R A_2^{(} \subseteq_R A_3^{(} \subseteq_R \dots \subseteq_R A_k^{(}$.

Then $A_1^{(} \vee_R A_2^{(} \vee_R A_3^{(} \vee_R \dots \vee_R A_k^{(} = A_k^{(}$.

Proof. Let

$$A_1^{(} = \langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \rangle, A_2^{(} = \langle [\tilde{A}_{2ij}^{a-}, \tilde{A}_{2ij}^{a+}], \lambda_{2ij}^a \rangle, A_3^{(} = \langle [\tilde{A}_{3ij}^{a-}, \tilde{A}_{3ij}^{a+}], \lambda_{3ij}^a \rangle, \dots,$$

$$A_k^{(} = \langle [\tilde{A}_{kij}^{a-}, \tilde{A}_{kij}^{a+}], \lambda_{kij}^a \rangle \in ICSM_{m \times n}^R.$$

Then, $A_1^{(} \vee_R A_2^{(} \vee_R A_3^{(} \vee_R \dots \vee_R A_k^{(} = (A_1^{(} \vee_R A_2^{(}) \vee_R A_3^{(} \vee_R \dots \vee_R A_k^{(} .$

Since, $A_1^{(} \subseteq_R A_2^{(} \Leftrightarrow \tilde{A}_{1,ij}^a \leq \tilde{A}_{2,ij}^a$ and $\lambda_{1,ij}^a \geq \lambda_{2,ij}^a$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} &= \left(\left\langle [\max\{\tilde{A}_{1,ij}^{a-}, \tilde{A}_{2,ij}^{a-}\}, \max\{\tilde{A}_{1,ij}^{a+}, \tilde{A}_{2,ij}^{a+}\}], \min\{\lambda_{1,ij}^a, \lambda_{2,ij}^a\} \right\rangle \right) \vee_R A_3^{(} \vee_R \dots \vee_R A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{2,ij}^{a-}, \tilde{A}_{2,ij}^{a+}], \lambda_{2,ij}^a \right\rangle \right) \vee_R A_3^{(} \vee_R \dots \vee_R A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{2,ij}^{a-}, \tilde{A}_{2,ij}^{a+}], \lambda_{2,ij}^a \right\rangle \vee_R \left\langle [\tilde{A}_{3,ij}^{a-}, \tilde{A}_{3,ij}^{a+}], \lambda_{3,ij}^a \right\rangle \right) \vee_R A_4^{(} \vee_R \dots \vee_R A_k^{(} . \end{aligned}$$

Since, $A_2^{(} \subseteq_R A_3^{(} \Leftrightarrow \tilde{A}_{2,ij}^a \leq \tilde{A}_{3,ij}^a$ and $\lambda_{2,ij}^a \geq \lambda_{3,ij}^a$ for all i, j .

Again by Definition 3.7(a),

$$\begin{aligned} &= \left(\left\langle [\max\{\tilde{A}_{2,ij}^{a-}, \tilde{A}_{3,ij}^{a-}\}, \max\{\tilde{A}_{2,ij}^{a+}, \tilde{A}_{3,ij}^{a+}\}], \min\{\lambda_{2,ij}^a, \lambda_{3,ij}^a\} \right\rangle \right) \vee_R A_4^{(} \vee_R \dots \vee_R A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{3,ij}^{a-}, \tilde{A}_{3,ij}^{a+}], \lambda_{3,ij}^a \right\rangle \right) \vee_R A_4^{(} \vee_R \dots \vee_R A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{3,ij}^{a-}, \tilde{A}_{3,ij}^{a+}], \lambda_{3,ij}^a \right\rangle \vee_R \left\langle [\tilde{A}_{4,ij}^{a-}, \tilde{A}_{4,ij}^{a+}], \lambda_{4,ij}^a \right\rangle \right) \vee_R A_5^{(} \vee_R \dots \vee_R A_k^{(} . \end{aligned}$$

Continuing the same process we get,

$$= \left(\left\langle [\tilde{A}_{(k-1),ij}^{a-}, \tilde{A}_{(k-1),ij}^{a+}], \lambda_{(k-1),ij}^a \right\rangle \vee_R \left\langle [\tilde{A}_{k,ij}^{a-}, \tilde{A}_{k,ij}^{a+}], \lambda_{k,ij}^a \right\rangle \right) .$$

Since, $A_{(k-1)}^{(} \subseteq_R A_k^{(} \Leftrightarrow \tilde{A}_{(k-1),ij}^a \leq \tilde{A}_{k,ij}^a$ and $\lambda_{(k-1),ij}^a \geq \lambda_{k,ij}^a$ for all i, j .

Again by Definition 3.7(a),

$$\begin{aligned} &= \left(\left\langle [\max\{\tilde{A}_{(k-1),ij}^{a-}, \tilde{A}_{k,ij}^{a-}\}, \max\{\tilde{A}_{(k-1),ij}^{a+}, \tilde{A}_{k,ij}^{a+}\}], \min\{\lambda_{(k-1),ij}^a, \lambda_{k,ij}^a\} \right\rangle \right) \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{k,ij}^{a-}, \tilde{A}_{k,ij}^{a+}], \lambda_{k,ij}^a \right\rangle \right) \\ &= \left(\left\langle \tilde{A}_{k,ij}^a, \lambda_{k,ij}^a \right\rangle \right) \\ &= A_k^{(} . \end{aligned}$$

Theorem 3.15 Let $A_1^{(}, A_2^{(}, A_3^{(}, \dots, A_k^{(} \in ICSM_{m \times n}^P$. Since $A_1^{(} \subseteq_P A_2^{(} \subseteq_P A_3^{(} \subseteq_P \dots \subseteq_P A_k^{(}$.

Then $A_1^{(} \wedge_P A_2^{(} \wedge_P A_3^{(} \wedge_P \dots \wedge_P A_k^{(} = A_1^{(}$.

Proof. Let

$$A_1^{(} = \left\langle [\tilde{A}_{1,ij}^{a-}, \tilde{A}_{1,ij}^{a+}], \lambda_{1,ij}^a \right\rangle, A_2^{(} = \left\langle [\tilde{A}_{2,ij}^{a-}, \tilde{A}_{2,ij}^{a+}], \lambda_{2,ij}^a \right\rangle, A_3^{(} = \left\langle [\tilde{A}_{3,ij}^{a-}, \tilde{A}_{3,ij}^{a+}], \lambda_{3,ij}^a \right\rangle, \dots,$$

$$A_k^{(} = \left\langle [\tilde{A}_{k,ij}^{a-}, \tilde{A}_{k,ij}^{a+}], \lambda_{k,ij}^a \right\rangle \in ICSM_{m \times n}^P .$$

Then, $A_1^{(} \wedge_P A_2^{(} \wedge_P A_3^{(} \wedge_P \dots \wedge_P A_k^{(} = (A_1^{(} \wedge_P A_2^{(}) \wedge_P A_3^{(} \wedge_P \dots \wedge_P A_k^{(} .$

Since, $A_1^{(} \subseteq_P A_2^{(} \Leftrightarrow \tilde{A}_{1,ij}^a \leq \tilde{A}_{2,ij}^a$ and $\lambda_{1,ij}^a \leq \lambda_{2,ij}^a$ for all i, j .

By Definition 3.5(b),

$$\begin{aligned} &= \left(\left\langle [\min\{\tilde{A}_{1,ij}^{a-}, \tilde{A}_{2,ij}^{a-}\}, \min\{\tilde{A}_{1,ij}^{a+}, \tilde{A}_{2,ij}^{a+}\}], \min\{\lambda_{1,ij}^a, \lambda_{2,ij}^a\} \right\rangle \right) \wedge_P A_3^{(} \wedge_P \dots \wedge_P A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{1,ij}^{a-}, \tilde{A}_{1,ij}^{a+}], \lambda_{1,ij}^a \right\rangle \right) \wedge_P A_3^{(} \wedge_P \dots \wedge_P A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{1,ij}^{a-}, \tilde{A}_{1,ij}^{a+}], \lambda_{1,ij}^a \right\rangle \wedge_P \left\langle [\tilde{A}_{3,ij}^{a-}, \tilde{A}_{3,ij}^{a+}], \lambda_{3,ij}^a \right\rangle \right) \wedge_P A_4^{(} \wedge_P \dots \wedge_P A_k^{(} . \end{aligned}$$

Since, $A_1^{(} \subseteq_P A_3^{(} \Leftrightarrow \tilde{A}_{1ij}^a \leq \tilde{A}_{3ij}^a$ and $\lambda_{1ij}^a \leq \lambda_{3ij}^a$ for all i, j .

Again by Definition 3.5(b),

$$\begin{aligned} &= \left(\left\langle [\min\{\tilde{A}_{1ij}^{a-}, \tilde{A}_{3ij}^{a-}\}, \min\{\tilde{A}_{1ij}^{a+}, \tilde{A}_{3ij}^{a+}\}], \min\{\lambda_{1ij}^a, \lambda_{3ij}^a\} \right\rangle \right) \wedge_P A_3^{(} \wedge_P \dots \wedge_P A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \right) \wedge_P A_4^{(} \wedge_P \dots \wedge_P A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \wedge_P \left\langle [\tilde{A}_{4ij}^{a-}, \tilde{A}_{4ij}^{a+}], \lambda_{4ij}^a \right\rangle \right) \wedge_P A_5^{(} \wedge_P \dots \wedge_P A_k^{(}. \end{aligned}$$

Continuing the same process we get,

$$= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \wedge_P \left\langle [\tilde{A}_{kij}^{a-}, \tilde{A}_{kij}^{a+}], \lambda_{kij}^a \right\rangle \right).$$

Since, $A_1^{(} \subseteq_P A_k^{(} \Leftrightarrow \tilde{A}_{1ij}^a \leq \tilde{A}_{kij}^a$ and $\lambda_{1ij}^a \leq \lambda_{kij}^a$ for all i, j .

Again by Definition 3.5(b),

$$\begin{aligned} &= \left(\left\langle [\min\{\tilde{A}_{1ij}^{a-}, \tilde{A}_{kij}^{a-}\}, \min\{\tilde{A}_{1ij}^{a+}, \tilde{A}_{kij}^{a+}\}], \min\{\lambda_{1ij}^a, \lambda_{kij}^a\} \right\rangle \right) \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \right) \\ &= \left(\left\langle \tilde{A}_{1ij}^a, \lambda_{1ij}^a \right\rangle \right) \\ &= A_1^{(}. \end{aligned}$$

Theorem 3.16 Let $A_1^{(}, A_2^{(}, A_3^{(}, \dots, A_k^{(} \in ICSM_{m \times n}^R$. Since $A_1^{(} \subseteq_R A_2^{(} \subseteq_R A_3^{(} \subseteq_R \dots \subseteq_R A_k^{(}$.

Then $A_1^{(} \wedge_R A_2^{(} \wedge_R A_3^{(} \wedge_R \dots \wedge_R A_k^{(} = A_1^{(}$.

Proof. Let

$$A_1^{(} = \left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle, A_2^{(} = \left\langle [\tilde{A}_{2ij}^{a-}, \tilde{A}_{2ij}^{a+}], \lambda_{2ij}^a \right\rangle, A_3^{(} = \left\langle [\tilde{A}_{3ij}^{a-}, \tilde{A}_{3ij}^{a+}], \lambda_{3ij}^a \right\rangle, \dots,$$

$$A_k^{(} = \left\langle [\tilde{A}_{kij}^{a-}, \tilde{A}_{kij}^{a+}], \lambda_{kij}^a \right\rangle \in ICSM_{m \times n}^R.$$

Then $A_1^{(} \wedge_R A_2^{(} \wedge_R A_3^{(} \wedge_R \dots \wedge_R A_k^{(} = (A_1^{(} \wedge_R A_2^{(}) \wedge_R A_3^{(} \wedge_R \dots \wedge_R A_k^{(}$.

Since, $A_1^{(} \subseteq_R A_2^{(} \Leftrightarrow \tilde{A}_{1ij}^a \leq \tilde{A}_{2ij}^a$ and $\lambda_{1ij}^a \geq \lambda_{2ij}^a$ for all i, j .

By Definition 3.7(b),

$$\begin{aligned} &= \left(\left\langle [\min\{\tilde{A}_{1ij}^{a-}, \tilde{A}_{2ij}^{a-}\}, \min\{\tilde{A}_{1ij}^{a+}, \tilde{A}_{2ij}^{a+}\}], \max\{\lambda_{1ij}^a, \lambda_{2ij}^a\} \right\rangle \right) \wedge_R A_3^{(} \wedge_R \dots \wedge_R A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \right) \wedge_R A_3^{(} \wedge_R \dots \wedge_R A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \wedge_R \left\langle [\tilde{A}_{3ij}^{a-}, \tilde{A}_{3ij}^{a+}], \lambda_{3ij}^a \right\rangle \right) \wedge_R A_4^{(} \wedge_R \dots \wedge_R A_k^{(}. \end{aligned}$$

Since, $A_1^{(} \subseteq_R A_3^{(} \Leftrightarrow \tilde{A}_{1ij}^a \leq \tilde{A}_{3ij}^a$ and $\lambda_{1ij}^a \geq \lambda_{3ij}^a$ for all i, j .

Again by Definition 3.7(b),

$$\begin{aligned} &= \left(\left\langle [\min\{\tilde{A}_{1ij}^{a-}, \tilde{A}_{3ij}^{a-}\}, \min\{\tilde{A}_{1ij}^{a+}, \tilde{A}_{3ij}^{a+}\}], \max\{\lambda_{1ij}^a, \lambda_{3ij}^a\} \right\rangle \right) \wedge_R A_3^{(} \wedge_R \dots \wedge_R A_k^{(} \text{ for all } i, j \\ &= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \right) \wedge_R A_4^{(} \wedge_R \dots \wedge_R A_k^{(} \\ &= \left(\left\langle [\tilde{A}_{1ij}^{a-}, \tilde{A}_{1ij}^{a+}], \lambda_{1ij}^a \right\rangle \wedge_R \left\langle [\tilde{A}_{4ij}^{a-}, \tilde{A}_{4ij}^{a+}], \lambda_{4ij}^a \right\rangle \right) \wedge_R A_5^{(} \wedge_R \dots \wedge_R A_k^{(}. \end{aligned}$$

Continuing the same process we get,

$$= \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{A}_{kij}^{a-}, \tilde{A}_{kij}^{a+}], \lambda_{kij}^a \right\rangle \right).$$

Since, $A_i^c \subseteq_R A_k^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{A}_{kij}^a$ and $\lambda_{ij}^a \geq \lambda_{kij}^a$ for all i, j.

Again by Definition 3.7(b),

$$\begin{aligned} &= \left(\left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{A}_{kij}^{a-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{A}_{kij}^{a+}\}], \max\{\lambda_{ij}^a, \lambda_{kij}^a\} \right\rangle \right) \text{ for all i, j} \\ &= \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \right) \\ &= \left(\left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \right) \\ &= A_i^c. \end{aligned}$$

Theorem 3.17 Let $A^c, B^c, C^c \in ICSM_{m \times n}^P$, then

$$(A^c \vee_P B^c) \wedge_P C^c = (A^c \wedge_P C^c) \vee_P (B^c \wedge_P C^c) = B^c.$$

Proof. Let $A^c = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle, B^c = \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle, C^c = \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \in ICSM_{m \times n}^P$.

$$\text{Then, L.H.S} = (A^c \vee_P B^c) \wedge_P C^c = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_P \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right) \wedge_P C^c.$$

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j.

By Definition 3.5(a),

$$\begin{aligned} (A^c \vee_P B^c) \wedge_P C^c &= \left(\left\langle [\max(\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}), \max(\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+})], \max(\lambda_{ij}^a, \mu_{ij}^b) \right\rangle \right) \wedge_P C^c \text{ for all i, j} \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \wedge_P C^c \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle. \end{aligned}$$

Since $B^c \subseteq_P C^c \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \leq \gamma_{ij}^c$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned} &= \left\langle [\min\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \min\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ &= B^c. \end{aligned}$$

$$\text{R.H.S} = (A^c \wedge_P C^c) \vee_P (B^c \wedge_P C^c).$$

$$\text{Then, } (A^c \wedge_P C^c) = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right).$$

Since $A^c \subseteq_P C^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned} (A^c \wedge_P C^c) &= \left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle. \end{aligned}$$

$$\text{Then, } (B^{\langle} \wedge_P C^{\langle}) = \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \right).$$

Since $B^{\langle} \subseteq_P C^{\langle} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \leq \gamma_{ij}^c$ for all i, j .

Again by Definition 3.5(b),

$$\begin{aligned} (B^{\langle} \wedge_P C^{\langle}) &= \left\langle [\min\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ (A^{\langle} \wedge_P C^{\langle}) \vee_P (B^{\langle} \wedge_P C^{\langle}) &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_P \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ &= A^{\langle} \vee_P B^{\langle}. \end{aligned}$$

Since $A^{\langle} \subseteq_P B^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} (A^{\langle} \wedge_P C^{\langle}) \vee_P (B^{\langle} \wedge_P C^{\langle}) &= \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \\ &= B^{\langle}. \end{aligned}$$

$$\text{Hence, } (A^{\langle} \vee_P B^{\langle}) \wedge_P C^{\langle} = (A^{\langle} \wedge_P C^{\langle}) \vee_P (B^{\langle} \wedge_P C^{\langle}) = B^{\langle}.$$

Theorem 3.18 Let $A^{\langle}, B^{\langle}, C^{\langle} \in ICSM_{m \times n}^R$, then

$$(A^{\langle} \vee_R B^{\langle}) \wedge_R C^{\langle} = (A^{\langle} \wedge_R C^{\langle}) \vee_R (B^{\langle} \wedge_R C^{\langle}) = B^{\langle}.$$

Proof. Let $A^{\langle} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, B^{\langle} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle, C^{\langle} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \in ICSM_{m \times n}^R$.

$$\text{Then, L.H.S} = (A^{\langle} \vee_R B^{\langle}) \wedge_R C^{\langle} = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \right) \wedge_R C^{\langle}.$$

Since $A^{\langle} \subseteq_R B^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} (A^{\langle} \vee_R B^{\langle}) \wedge_R C^{\langle} &= \left(\left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right) \wedge_R C^{\langle} \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \wedge_R C^{\langle} \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle. \end{aligned}$$

Since $B^{\langle} \subseteq_R C^{\langle} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j .

By Definition 3.7(b),

$$\begin{aligned} &= \left\langle [\min\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ &= B^{\langle}. \end{aligned}$$

$$\text{R.H.S} = (A^{\langle} \wedge_R C^{\langle}) \vee_R (B^{\langle} \wedge_R C^{\langle})$$

Then, $(A^{\langle} \wedge_R C^{\langle}) = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right)$.

Since $A^{\langle} \subseteq_R C^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned} (A^{\langle} \wedge_R C^{\langle}) &= \left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle. \end{aligned}$$

Then, $(B^{\langle} \wedge_R C^{\langle}) = \left(\left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right)$.

Since $B^{\langle} \subseteq_R C^{\langle} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j.

Again by Definition 3.7(b),

$$\begin{aligned} (B^{\langle} \wedge_R C^{\langle}) &= \left\langle [\min\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle. \\ (A^{\langle} \wedge_R C^{\langle}) \vee_R (B^{\langle} \wedge_R C^{\langle}) &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_R \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ &= A^{\langle} \vee_R B^{\langle}. \end{aligned}$$

Since $A^{\langle} \subseteq_R B^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned} (A^{\langle} \wedge_R C^{\langle}) \vee_R (B^{\langle} \wedge_R C^{\langle}) &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= B^{\langle}. \end{aligned}$$

$$\text{Hence, } (A^{\langle} \vee_R B^{\langle}) \wedge_R C^{\langle} = (A^{\langle} \wedge_R C^{\langle}) \vee_R (B^{\langle} \wedge_R C^{\langle}) = B^{\langle}.$$

Theorem 3.19 Let $A^{\langle}, B^{\langle}, C^{\langle} \in ICSM_{m \times n}^P$, then

$$(A^{\langle} \wedge_P B^{\langle}) \vee_P C^{\langle} = (A^{\langle} \vee_P C^{\langle}) \wedge_P (B^{\langle} \vee_P C^{\langle}) = C^{\langle}.$$

Proof. Let $A^{\langle} = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle, B^{\langle} = \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle, C^{\langle} = \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \in ICSM_{m \times n}^P$.

Then, L.H.S = $(A^{\langle} \wedge_P B^{\langle}) \vee_P C^{\langle} = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right) \vee_P C^{\langle}$.

Since $A^{\langle} \subseteq_P B^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned} (A^{\langle} \wedge_P B^{\langle}) \vee_P C^{\langle} &= \left(\left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right) \vee_P C^{\langle} \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_P C^{\langle} \\ &= \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle. \end{aligned}$$

Since $A^{\langle} \subseteq_P C^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j .

By Definition 3.5(b),

$$\begin{aligned} &= \left\langle [\max(\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}), \min(\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+})], \min(\lambda_{ij}^a, \gamma_{ij}^c) \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= C^{\langle}. \end{aligned}$$

$$\text{R.H.S} = (A^{\langle} \vee_P C^{\langle}) \wedge_P (B^{\langle} \vee_P C^{\langle})$$

$$\text{Then, } (A^{\langle} \vee_P C^{\langle}) = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right).$$

Since $A^{\langle} \subseteq_P C^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} (A^{\langle} \vee_P C^{\langle}) &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle. \end{aligned}$$

$$\text{Then, } (B^{\langle} \vee_P C^{\langle}) = \left(\left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right).$$

Since $B^{\langle} \subseteq_P C^{\langle} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \leq \gamma_{ij}^c$ for all i, j .

Again by Definition 3.5(a),

$$\begin{aligned} (B^{\langle} \vee_P C^{\langle}) &= \left\langle [\max\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ (A^{\langle} \vee_P C^{\langle}) \wedge_P (B^{\langle} \vee_P C^{\langle}) &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \wedge_P \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= C^{\langle} \wedge_P C^{\langle}. \end{aligned}$$

By Proposition 3.11(i),

$$(A^{\langle} \vee_P C^{\langle}) \wedge_P (B^{\langle} \vee_P C^{\langle}) = C^{\langle}. \text{ Hence, } (A^{\langle} \wedge_P B^{\langle}) \vee_P C^{\langle} = (A^{\langle} \vee_P C^{\langle}) \wedge_P (B^{\langle} \vee_P C^{\langle}) = C^{\langle}.$$

Theorem 3.20 Let $A^{\langle}, B^{\langle}, C^{\langle} \in ICSM_{m \times n}^R$, then

$$(A^{\langle} \wedge_R B^{\langle}) \vee_R C^{\langle} = (A^{\langle} \vee_R C^{\langle}) \wedge_R (B^{\langle} \vee_R C^{\langle}) = C^{\langle}.$$

Proof. Let $A^{\langle} = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle, B^{\langle} = \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle, C^{\langle} = \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \in ICSM_{m \times n}^R$.

$$\text{Then, L.H.S} = (A^{\langle} \wedge_R B^{\langle}) \vee_R C^{\langle} = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right) \vee_R C^{\langle}.$$

Since $A^{\langle} \subseteq_R B^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(b),

$$\begin{aligned} (A^{\langle} \wedge_R B^{\langle}) \vee_R C^{\langle} &= \left(\left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right) \vee_R C^{\langle} \text{ for all } i, j \\ &= \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_R C^{\langle} \\ &= \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle. \end{aligned}$$

Since $A^{\langle} \subseteq_R C^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= C^{\langle}. \end{aligned}$$

$$\text{R.H.S} = (A^{\langle} \vee_R C^{\langle}) \wedge_R (B^{\langle} \vee_R C^{\langle}).$$

$$\text{Then, } (A^{\langle} \vee_R C^{\langle}) = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right).$$

Since $A^{\langle} \subseteq_R C^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned} (A^{\langle} \vee_R C^{\langle}) &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle. \end{aligned}$$

$$\text{Then, } (B^{\langle} \vee_R C^{\langle}) = \left(\left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \vee_R \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right).$$

Since $B^{\langle} \subseteq_R C^{\langle} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j .

Again by Definition 3.7(a),

$$\begin{aligned} (B^{\langle} \vee_R C^{\langle}) &= \left\langle [\max\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \min\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle. \\ (A^{\langle} \vee_R C^{\langle}) \wedge_R (B^{\langle} \vee_R C^{\langle}) &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \wedge_R \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= C^{\langle} \wedge_R C^{\langle}. \end{aligned}$$

By Proposition 3.12(i), $(A^{\langle} \vee_R C^{\langle}) \wedge_R (B^{\langle} \vee_R C^{\langle}) = C^{\langle}$.

Hence, $(A^{\langle} \wedge_R B^{\langle}) \vee_R C^{\langle} = (A^{\langle} \vee_R C^{\langle}) \wedge_R (B^{\langle} \vee_R C^{\langle}) = C^{\langle}$.

Theorem 3.21 Let $A^{\langle}, B^{\langle}, C^{\langle} \in ICSM_{m \times n}^P$, then

$$A^{\langle} \vee_P (B^{\langle} \wedge_P C^{\langle}) = (A^{\langle} \vee_P B^{\langle}) \wedge_P (A^{\langle} \vee_P C^{\langle}) = B^{\langle}.$$

Proof. Let $A^{\langle} = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle, B^{\langle} = \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle, C^{\langle} = \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \in ICSM_{m \times n}^P$.

$$\text{Then, L.H.S} = A^{\langle} \vee_P (B^{\langle} \wedge_P C^{\langle}) = A^{\langle} \vee_P \left(\left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \right).$$

Since $B^{\langle} \subseteq_P C^{\langle} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \leq \gamma_{ij}^c$ for all i, j .

By Definition 3.5(b),

$$\begin{aligned} A^{\langle} \vee_P (B^{\langle} \wedge_P C^{\langle}) &= A^{\langle} \vee_P \left(\left\langle [\min(\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}), \min(\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+})], \min(\mu_{ij}^b, \gamma_{ij}^c) \right\rangle \right) \text{ for all } i, j \\ &= A^{\langle} \vee_P \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_P \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle. \end{aligned}$$

Since $A^{\langle} \subseteq_p B^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ &= B^{\langle}. \end{aligned}$$

$$\text{R.H.S} = (A^{\langle} \vee_p B^{\langle}) \wedge_p (A^{\langle} \vee_p C^{\langle}).$$

$$\text{Then, } (A^{\langle} \vee_p B^{\langle}) = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_p \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle.$$

Since $A^{\langle} \subseteq_p B^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j .

By Definition 3.5(a),

$$\begin{aligned} (A^{\langle} \vee_p B^{\langle}) &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle. \end{aligned}$$

$$\text{Then, } (A^{\langle} \vee_p C^{\langle}) = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_p \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle.$$

Since $A^{\langle} \subseteq_p C^{\langle} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j .

Again by Definition 3.5(a),

$$\begin{aligned} (A^{\langle} \vee_p C^{\langle}) &= \left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ (A^{\langle} \vee_p B^{\langle}) \wedge_p (A^{\langle} \vee_p C^{\langle}) &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \wedge_p \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= B^{\langle} \wedge_p C^{\langle}. \end{aligned}$$

Since $B^{\langle} \subseteq_p C^{\langle} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \leq \gamma_{ij}^c$ for all i, j .

By Definition 3.5(b),

$$\begin{aligned} (A^{\langle} \vee_p B^{\langle}) \wedge_p (A^{\langle} \vee_p C^{\langle}) &= \left\langle [\min\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \min\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all } i, j \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \\ &= B^{\langle}. \end{aligned}$$

$$\text{Hence, } A^{\langle} \vee_p (B^{\langle} \wedge_p C^{\langle}) = (A^{\langle} \vee_p B^{\langle}) \wedge_p (A^{\langle} \vee_p C^{\langle}) = B^{\langle}.$$

Theorem 3.22 Let $A^{\langle}, B^{\langle}, C^{\langle} \in ICSM_{m \times n}^R$, then

$$A^{\langle} \vee_R (B^{\langle} \wedge_R C^{\langle}) = (A^{\langle} \vee_R B^{\langle}) \wedge_R (A^{\langle} \vee_R C^{\langle}) = B^{\langle}.$$

Proof. Let $A^{\langle} = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle, B^{\langle} = \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle, C^{\langle} = \left\langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \right\rangle \in ICSM_{m \times n}^R$.

Then, L.H.S = $A^{(} \vee_R (B^{(} \wedge_R C^{(}) = A^{(} \vee_R \left(\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \wedge_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \right)$.

Since $B^{(} \subseteq_R C^{(} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned} A^{(} \vee_R (B^{(} \wedge_R C^{(}) &= A^{(} \vee_R \left(\langle [\min(\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}), \min(\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+})], \max(\mu_{ij}^b, \gamma_{ij}^c) \rangle \right) \text{ for all i, j} \\ &= A^{(} \vee_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle. \end{aligned}$$

Since $A^{(} \subseteq_R B^{(} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned} &= \langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \\ &= \langle \tilde{B}_{ij}^b, \mu_{ij}^b \rangle \\ &= B^{(}. \end{aligned}$$

R.H.S = $(A^{(} \vee_R B^{(}) \wedge_R (A^{(} \vee_R C^{(})$

Then, $(A^{(} \vee_R B^{(}) = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right)$.

Since $A^{(} \subseteq_R B^{(} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned} (A^{(} \vee_R B^{(}) &= \langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \\ &= \langle \tilde{B}_{ij}^b, \mu_{ij}^b \rangle. \end{aligned}$$

Then, $(A^{(} \vee_R C^{(}) = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \right)$.

Since $A^{(} \subseteq_R C^{(} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j. Again by Definition 3.7(a),

$$\begin{aligned} (A^{(} \vee_R C^{(}) &= \langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \\ &= \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle \\ (A^{(} \vee_R B^{(}) \wedge_R (A^{(} \vee_R C^{(}) &= \langle \tilde{B}_{ij}^b, \mu_{ij}^b \rangle \wedge_R \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle \\ &= B^{(} \wedge_R C^{(}. \end{aligned}$$

Since $B^{(} \subseteq_R C^{(} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned} (A^{(} \vee_R B^{(}) \wedge_R (A^{(} \vee_R C^{(}) &= \langle [\min\{\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+}\}], \max\{\mu_{ij}^b, \gamma_{ij}^c\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \\ &= \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \\ &= B^{(}. \end{aligned}$$

Hence, $A^c \vee_R (B^c \wedge_R C^c) = (A^c \vee_R B^c) \wedge_R (A^c \vee_R C^c) = B^c$.

Theorem 3.23 Let $A^c, B^c, C^c \in ICSM_{m \times n}^P$, then

$$A^c \wedge_P (B^c \vee_P C^c) = (A^c \wedge_P B^c) \vee_P (A^c \wedge_P C^c) = A^c.$$

Proof. Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle, B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle, C^c = \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \in ICSM_{m \times n}^P$.

Then, L.H.S = $A^c \wedge_P (B^c \vee_P C^c) = A^c \wedge_P \left(\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \vee_P \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \right)$.

Since $B^c \subseteq_P C^c \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \leq \gamma_{ij}^c$ for all i, j. By Definition 3.5(a),

$$\begin{aligned} A^c \wedge_P (B^c \vee_P C^c) &= A^c \wedge_P \left(\langle [\max(\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}), \max(\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+})], \max(\mu_{ij}^b, \gamma_{ij}^c) \rangle \right) \text{ for all i, j} \\ &= A^c \wedge_P \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle. \end{aligned}$$

Since $A^c \subseteq_P C^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j. By Definition 3.5(b),

$$\begin{aligned} &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

R.H.S = $(A^c \wedge_P B^c) \vee_P (A^c \wedge_P C^c)$.

Then, $(A^c \wedge_P B^c) = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right)$.

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j. By Definition 3.5(b),

$$\begin{aligned} (A^c \wedge_P B^c) &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle. \end{aligned}$$

Then, $(A^c \wedge_P C^c) = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \right)$.

Since $A^c \subseteq_P C^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j.

Again by Definition 3.5(b),

$$\begin{aligned} (A^c \wedge_P C^c) &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ (A^c \wedge_P B^c) \vee_P (A^c \wedge_P C^c) &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \vee_P \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c \vee_P A^c. \end{aligned}$$

By Proposition 3.9(i), $(A^c \wedge_P C^c) \vee_P (B^c \wedge_P C^c) = A^c$.

Hence, $A^c \wedge_P (B^c \vee_P C^c) = (A^c \wedge_P B^c) \vee_P (A^c \wedge_P C^c) = A^c$.

Theorem 3.24 Let $A^c, B^c, C^c \in ICSM_{m \times n}^R$, then

$$A^c \wedge_R (B^c \vee_R C^c) = (A^c \wedge_R B^c) \vee_R (A^c \wedge_R C^c) = A^c.$$

Proof. Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle, B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle, C^c = \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \in ICSM_{m \times n}^R$.

$$\text{Then, L.H.S} = A^c \wedge_R (B^c \vee_R C^c) = A^c \wedge_R \left(\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \vee_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \right).$$

Since $B^c \subseteq_R C^c \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned} A^c \wedge_R (B^c \vee_R C^c) &= A^c \wedge_R \left(\langle [\max(\tilde{B}_{ij}^{b-}, \tilde{C}_{ij}^{c-}), \max(\tilde{B}_{ij}^{b+}, \tilde{C}_{ij}^{c+})], \min(\mu_{ij}^b, \gamma_{ij}^c) \rangle \right) \text{ for all i, j} \\ &= A^c \wedge_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle. \end{aligned}$$

Since $A^c \subseteq_R C^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned} &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c. \end{aligned}$$

$$\text{R.H.S} = (A^c \wedge_R B^c) \vee_R (A^c \wedge_R C^c).$$

$$\text{Then, } (A^c \wedge_R B^c) = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right).$$

Since $A^c \subseteq_R B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned} (A^c \wedge_R B^c) &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle. \end{aligned}$$

$$\text{Then, } (A^c \wedge_R C^c) = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_R \langle [\tilde{C}_{ij}^{c-}, \tilde{C}_{ij}^{c+}], \gamma_{ij}^c \rangle \right).$$

Since $A^c \subseteq_R C^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j.

Again by Definition 3.7(b),

$$\begin{aligned} (A^c \wedge_R C^c) &= \langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{C}_{ij}^{c-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{C}_{ij}^{c+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \rangle \text{ for all i, j} \\ &= \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \\ &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ (A^c \wedge_R B^c) \vee_R (A^c \wedge_R C^c) &= \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \vee_R \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \\ &= A^c \vee_R A^c. \end{aligned}$$

By Proposition 3.10(i), $(A^c \wedge_R C^c) \vee_R (B^c \wedge_R C^c) = A^c$.

Hence, $A^c \wedge_R (B^c \vee_R C^c) = (A^c \wedge_R B^c) \vee_R (A^c \wedge_R C^c) = A^c$.

Theorem 3.25 Let $A^c, B^c \in ICSM_{m \times n}^P$. Then

(i) $(A^c \vee_P B^c)^c = B^{c^c} \wedge_P A^{c^c}$.

(ii) $(A^c \wedge_P B^c)^c = B^{c^c} \vee_P A^{c^c}$.

Proof. Let $A^c = \langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle$ and $B^c = \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle$.

(i) L.H.S = $(A^c \vee_P B^c)^c = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \vee_P \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right)^c$.

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j.

By the Definition 3.5(a),

$$\begin{aligned} &= \left(\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \right)^c \text{ for all i, j} \\ &= \langle [1 - \max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, 1 - \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], 1 - \max\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \\ &= \langle [1 - \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}, 1 - \max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}], 1 - \max\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \\ &= \langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \rangle \\ &= C^c. \end{aligned}$$

R.H.S = $B^{c^c} \wedge_P A^{c^c}$

$$\begin{aligned} &= \left(\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right)^c \wedge_P \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \right)^c \\ &= \langle [1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{B}_{ij}^{b+}], 1 - \mu_{ij}^b \rangle \wedge_P \langle [1 - \tilde{A}_{ij}^{a-}, 1 - \tilde{A}_{ij}^{a+}], 1 - \lambda_{ij}^a \rangle \\ &= \langle [1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{B}_{ij}^{b-}], 1 - \mu_{ij}^b \rangle \wedge_P \langle [1 - \tilde{A}_{ij}^{a+}, 1 - \tilde{A}_{ij}^{a-}], 1 - \lambda_{ij}^a \rangle. \end{aligned}$$

Since $B^{c^c} \subseteq_P A^{c^c}$, by Definition 3.5(b),

$$\begin{aligned} &= \langle [\min\{1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{A}_{ij}^{a+}\}, \min\{1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{A}_{ij}^{a-}\}], \min\{1 - \mu_{ij}^b, 1 - \lambda_{ij}^a\} \rangle \\ &= \langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \rangle \\ &= C^c. \end{aligned}$$

L.H.S = R.H.S

So $(A^c \vee_P B^c)^c = B^{c^c} \wedge_P A^{c^c}$.

(ii) L.H.S = $(A^c \wedge_P B^c)^c = \left(\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \rangle \wedge_P \langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \rangle \right)^c$.

Since $A^c \subseteq_P B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned} &= \left(\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \right)^c \text{ for all i, j} \\ &= \langle [1 - \min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, 1 - \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], 1 - \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \end{aligned}$$

$$\begin{aligned}
 &= \left\langle [1 - \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}, 1 - \min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}], 1 - \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \\
 &= \left\langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \right\rangle \\
 &= C^c.
 \end{aligned}$$

$$\text{R.H.S} = B^{(c)} \vee_P A^{(c)}$$

$$\begin{aligned}
 &= \left(\left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right)^c \vee_P \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \right)^c \\
 &= \left\langle [1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{B}_{ij}^{b+}], 1 - \mu_{ij}^b \right\rangle \vee_P \left\langle [1 - \tilde{A}_{ij}^{a-}, 1 - \tilde{A}_{ij}^{a+}], 1 - \lambda_{ij}^a \right\rangle \\
 &= \left\langle [1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{B}_{ij}^{b-}], 1 - \mu_{ij}^b \right\rangle \vee_P \left\langle [1 - \tilde{A}_{ij}^{a+}, 1 - \tilde{A}_{ij}^{a-}], 1 - \lambda_{ij}^a \right\rangle.
 \end{aligned}$$

Since $B^{(c)} \subseteq_P A^{(c)}$, by Definition 3.5(a),

$$\begin{aligned}
 &= \left\langle [\max\{1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{A}_{ij}^{a+}\}, \max\{1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{A}_{ij}^{a-}\}], \max\{1 - \mu_{ij}^b, 1 - \lambda_{ij}^a\} \right\rangle \\
 &= \left\langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \right\rangle \\
 &= C^c.
 \end{aligned}$$

L.H.S = R.H.S

So $(A^c \wedge_P B^c)^c = B^{(c)} \vee_P A^{(c)}$.

Theorem 3.26 Let $A^c, B^c \in ICSM_{m \times n}^R$. Then

(i) $(A^c \vee_R B^c)^c = B^{(c)} \wedge_R A^{(c)}$.

(ii) $(A^c \wedge_R B^c)^c = B^{(c)} \vee_R A^{(c)}$.

Proof. Let $A^c = \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle$ and $B^c = \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle$.

(i) L.H.S = $(A^c \vee_R B^c)^c = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right)^c$.

Since $A^c \subseteq_R B^c \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(a),

$$\begin{aligned}
 &= \left(\left\langle [\max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right)^c \text{ for all } i, j \\
 &= \left\langle [1 - \max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, 1 - \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], 1 - \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \\
 &= \left\langle [1 - \max\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}, 1 - \max\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}], 1 - \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \\
 &= \left\langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \right\rangle \\
 &= C^c.
 \end{aligned}$$

$$\text{R.H.S} = B^{(c)} \wedge_R A^{(c)}$$

$$\begin{aligned}
 &= \left(\left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right)^c \wedge_R \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \right)^c \\
 &= \left\langle [1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{B}_{ij}^{b+}], 1 - \mu_{ij}^b \right\rangle \wedge_R \left\langle [1 - \tilde{A}_{ij}^{a-}, 1 - \tilde{A}_{ij}^{a+}], 1 - \lambda_{ij}^a \right\rangle
 \end{aligned}$$

$$= \left\langle [1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{B}_{ij}^{b-}], 1 - \mu_{ij}^b \right\rangle \wedge_R \left\langle [1 - \tilde{A}_{ij}^{a+}, 1 - \tilde{A}_{ij}^{a-}], 1 - \lambda_{ij}^a \right\rangle.$$

Since $B^{(c)} \subseteq_R A^{(c)}$, by Definition 3.7(b),

$$\begin{aligned} &= \left\langle [\min\{1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{A}_{ij}^{a+}\}, \min\{1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{A}_{ij}^{a-}\}], \max\{1 - \mu_{ij}^b, 1 - \lambda_{ij}^a\} \right\rangle \\ &= \left\langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \right\rangle \\ &= C^{(c)}. \end{aligned}$$

L.H.S = R.H.S

So $(A^{(c)} \vee_R B^{(c)})^c = B^{(c)} \wedge_R A^{(c)}$.

$$(ii) \quad \text{L.H.S} = (A^{(c)} \wedge_R B^{(c)})^c = \left(\left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \right)^c.$$

Since $A^{(c)} \subseteq_R B^{(c)} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j .

By Definition 3.7(b),

$$\begin{aligned} &= \left(\left\langle [\min\{\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}\}, \min\{\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right)^c \text{ for all } i, j \\ &= \left\langle [1 - \min(\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-}), 1 - \min(\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+})], 1 - \max(\lambda_{ij}^a, \mu_{ij}^b) \right\rangle \\ &= \left\langle [1 - \min(\tilde{A}_{ij}^{a+}, \tilde{B}_{ij}^{b+}), 1 - \min(\tilde{A}_{ij}^{a-}, \tilde{B}_{ij}^{b-})], 1 - \max(\lambda_{ij}^a, \mu_{ij}^b) \right\rangle \\ &= \left\langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \right\rangle \\ &= C^{(c)}. \end{aligned}$$

R.H.S = $B^{(c)} \vee_R A^{(c)}$

$$\begin{aligned} &= \left(\left\langle [\tilde{B}_{ij}^{b-}, \tilde{B}_{ij}^{b+}], \mu_{ij}^b \right\rangle \vee_R \left\langle [\tilde{A}_{ij}^{a-}, \tilde{A}_{ij}^{a+}], \lambda_{ij}^a \right\rangle \right)^c \\ &= \left\langle [1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{B}_{ij}^{b+}], 1 - \mu_{ij}^b \right\rangle \vee_R \left\langle [1 - \tilde{A}_{ij}^{a-}, 1 - \tilde{A}_{ij}^{a+}], 1 - \lambda_{ij}^a \right\rangle \\ &= \left\langle [1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{B}_{ij}^{b-}], 1 - \mu_{ij}^b \right\rangle \vee_R \left\langle [1 - \tilde{A}_{ij}^{a+}, 1 - \tilde{A}_{ij}^{a-}], 1 - \lambda_{ij}^a \right\rangle. \end{aligned}$$

Since $B^{(c)} \subseteq_R A^{(c)}$, by Definition 3.7(a),

$$\begin{aligned} &= \left\langle [\max\{1 - \tilde{B}_{ij}^{b+}, 1 - \tilde{A}_{ij}^{a+}\}, \max\{1 - \tilde{B}_{ij}^{b-}, 1 - \tilde{A}_{ij}^{a-}\}], \min\{1 - \mu_{ij}^b, 1 - \lambda_{ij}^a\} \right\rangle \\ &= \left\langle [\tilde{C}_{ij}^{c+}, \tilde{C}_{ij}^{c-}], \gamma_{ij}^c \right\rangle \\ &= C^{(c)}. \end{aligned}$$

L.H.S = R.H.S

So $(A^{(c)} \wedge_R B^{(c)})^c = B^{(c)} \vee_R A^{(c)}$.

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