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SPLIT AND NON SPLIT TWO DOMINATION NUMBER OF A GRAPH

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Abstract:

A Subset D of V is called a dominating set in G if every vertex in V-D is adjacent to at least one vertex in S. A Dominating set is said to be two dominating set if every vertex in V-D is adjacent to atleast two vertices in D. The minimum cardinality taken over all, the minimal two dominating set is called two domination number and is denoted by $\gamma_2(G)$. In this paper, we introduce the concept of split two domination number and non split two domination number of a graph. A Dominating set $D \subseteq V$ of a graph G is a Split two dominating set if the induced subgraph <V-D> is disconnected and every vertex in V-D is adjacent to atleast two vertices in D. The Split two domination number γ_{s2} (G) is the minimum cardinality of a Split two dominating set. A Dominating set $D \subseteq V$ of a graph G is a Non Split two dominating set if the induced subgraph <V-D> is connected and every vertex in V-D is adjacent to atleast two vertices in D. The Non Split two domination number γ_{ns2} (G) is the minimum cardinality of a Non Split two dominating set. We found this parameter for some standard classes of graphs also we obtain some bounds for the above said parameter.

Key Words: Split Two Domination Number & Non Split Two Domination Number

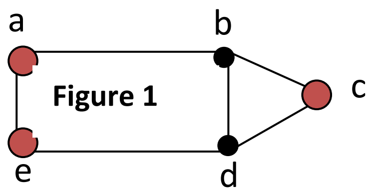
1. Introduction:

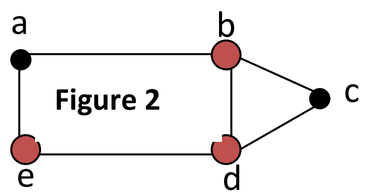
Let G = (V, E) be a simple undirected graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by d(u).The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively. P_n denotes the path on n vertices. The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph. A subset S of V is called a dominating set in G if every vertex in V-S is adjacent to atleast one vertex in S. The minimum cardinality taken over all dominating sets in G is called the domination number of G and is denoted byγ.A dominating set is said to be two dominating set if every vertex in V-S is adjacent to atleast two vertices in S. The minimum cardinality taken over all the minimal two dominating set is called two domination number and is denoted by $\gamma_2(G)$. A Dominating set $D \subseteq V$ of a graph G is a Split (Non Split) two dominating set if the induced subgraph <V-D> is disconnected (connected) and every vertex in V-D is adjacent to atleast two vertices in D. The Split (Non Split) two domination number γ_{s2} (G) (γ_{ns2} (G)) is the minimum cardinality of a Split (Non Split) two dominating set.

1.1 Notations:

A **triangular snake** graph is obtained from a path $v_1, v_2, ...v_n$ by joining V_i and V_{i+1} to a new vertex w_i for i = 1,2,..., n-1 and denoted by mC_3-snake. Any cycle with a pendant edge attached at each vertex is called **crown graph** and is denoted by C_n^+. Any path with a pendant edge attached at each vertex is called **Hoffman tree** and is denoted by P_n^+. Any wheel with a pendant edge attached at each vertex is denoted by W_n^+. Any complete graph with a pendant edge attached at each vertex is K_n^+. Any fan with a pendant edge attached at each vertex is denoted by F_n^+. The **Book** B_n is the graph $S_n \times P_m$ where S_n is the star with n+1vertices. The graph $C_n^{(t)}$ denote the one point union of t cycles of length n. If n=3, it is called the **Dutch t-windmill or friendship** graph. The graph $P_n^{(t)}$ denote the one point union of t paths of length n.$K_n(P_m)$ denotes the graph obtained from K_n by attaching the end vertex of P_m to anyone vertex of K_n. $K_n(m_1,m_2,m_3,…..m_k)$ denotes the graph obtained from K_n by attaching m_1 edges to the vertex u_i of K_n,m_2 edges to the vertex u_j for i≠j of K_n ……….,m_k edges to all the distinct vertices of K_n. $S(K_{1,m})$ is a graph obtained from $K_{1,m}$ by subdividing ant one edge of $K_{1,m}$. The **Corona** of two graphs G×H is defined as, by taking one copy of G and taking $|G|$ copies of H and join every i^{th} vertex of G to all vertices of different copies of H.

1.2 Example:

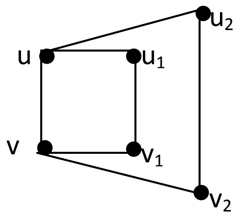




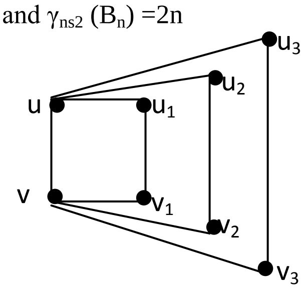
In figure 1, D= { a,c,e } and V-D = {b,d}. Hence <V-D> is connected, γ_{ns2} (G) = 3.

In figure 2, D= { b,d,e } and V-D = {a,c}. Hence <V-D> is disconnected, γ_{s2} (G) = 3.

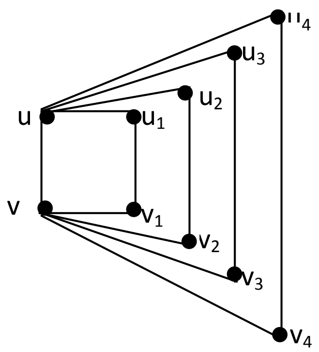
- ✓ For all paths, γ_{ns2} (P_n) = n-2 for n≥ 4 and
  γ_{s2} (P_n) = n+1/2, if n is odd
  n+2/2 , if n is even
- ✓ For all cycles, γ_{ns2} (C_n) = n-2 for n≥ 6 and
  γ_{s2} (C_n) = n+1/2, if n is odd
  n/2 , if n is even
- ✓ For all complete graphs,γ_{ns2} (K_n) =2, for n≥ 3
- ✓ For all wheel graphs, γ_{ns2} (W_n) = n-1, for n ≥ 5and
  γ_{s2} (W_n) = n/2 , if n is even, for n≥ 6
  n+1/2 , if n is odd, for n≥ 5
- ✓ For all complete bipartite graphs,γ_{s2} $(K_{m,n})$ = min { m,n }, for m,n ≥ 2
- ✓ For all star graphs, γ_{ns2} $(K_{1,n})$ = n and γ_{s2} $(K_{1,n})$ = n for all n≥ 2.
- ✓ For a bistar, γ_{s2} $(B_{m,n})$ = m+n-1, γ_{ns2} $(B_{m,n})$ = m+n-2.
- ✓ For all trees, γ_{ns2} (T) = n-2for all n ≥ 5 and γ_{s2} (T) = n-1 for all n ≥ 3.
- ✓ γ_{ns2} (K_n^+) = n+2, for n ≥ 3
- ✓ For all book graphs, γ_{s2} (B_n) = n+1 for all n ≥ 2 and γ_{ns2} (B_n) =2n



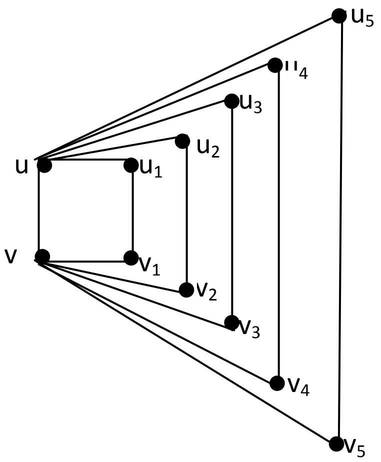
γ_{s2} (B_2) = 3
γ_{ns2} (B_2) = 4



γ_{s2} (B_3) = 4
γ_{ns2} (B_3) = 6

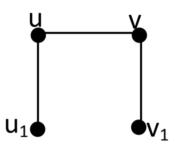


γ_{s2} (B_4) = 5
γ_{ns2} (B_4) = 8

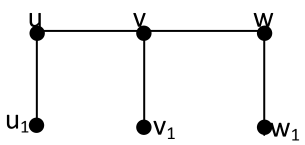


γ_{s2} (B_5) = 6
γ_{ns2} (B_5) = 10

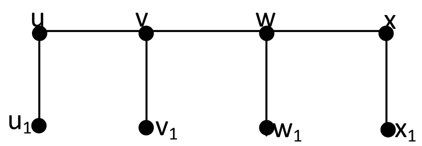
- ✓ For all Hoffman trees, γ_{ns2} (P_n^+) = 2n-2 if n is even
  2n-1 if n is odd for n>2
  γ_{s2} (P_n^+) = n+2 if n is even
  2n-2 if n is odd for n>2



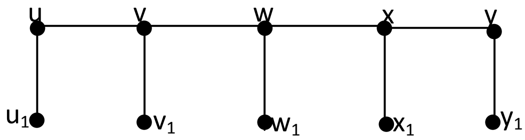
γ_{s2} (P_2^+) = 3
γ_{ns2} (P_2^+) = 3



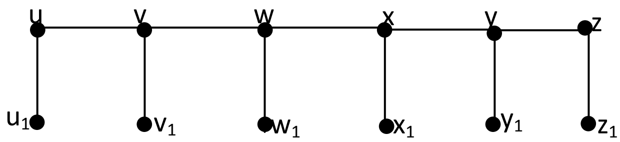
γ_{s2} (P_3^+) = 4
γ_{ns2} (P_3^+) = 5



γ_{s2} (P_4^+) = 6
γ_{ns2} (P_4^+) = 6

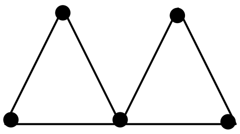


γ_{s2} (P_5') = 8
γ_{s2} (P_5^+) = 9

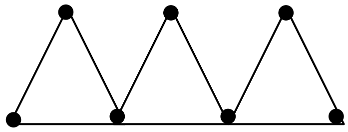


γ_{s2} (P_6') = 8
γ_{s2} (P_6^+) = 10

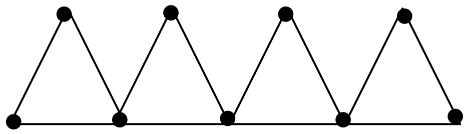
✓ For all triangular snake, γ_{ns2} (mC_3) = m+2, for n≥ 2 and γ_{s2} (mC_3) = m+1, for n≥ 2



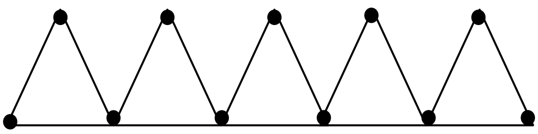
γ_{s2} ($2C_3$) =3
γ_{ns2} ($2C_3$) =4



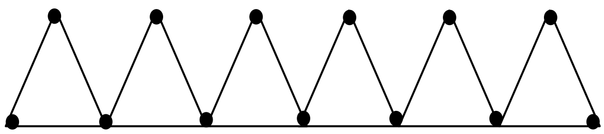
γ_{s2} ($3C_3$) =4
γ_{ns2} ($3C_3$) =6



γ_{s2} ($4C_3$) =5
γ_{ns2} ($4C_3$) =6

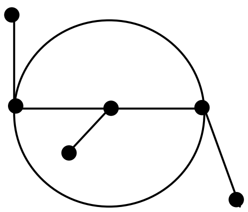


γ_{s2} ($5C_3$) =6
γ_{ns2} ($5C_3$) =7

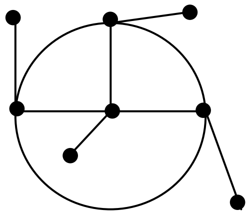


γ_{s2} ($6C_3$) =7
γ_{ns2} ($6C_3$) =8

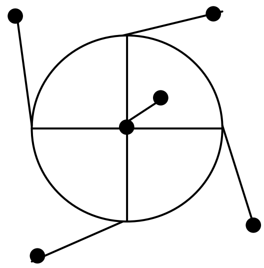
✓ γ_{ns2} (W_n^+) = n+2 , for n ≥ 3 and γ_{s2} (W_n^+) = n+4, for n≥3



γ_{s2} (W_2^+) = 5
γ_{ns2} (W_2^+) = 4

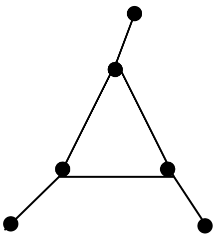


γ_{s2} (W_3^+) = 7
γ_{ns2} (W_3^+) =5

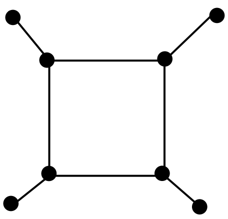


γ_{s2} (W_4^+) = 8
γ_{ns2} (W_4^+) = 6

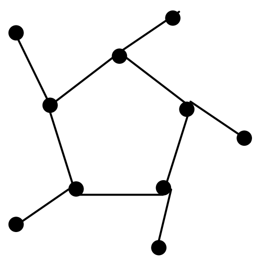
✓ For all crown graphs, γ_{ns2} (C_n^+) = 2n-2 and γ_{s2} (C_n^+) = n+2



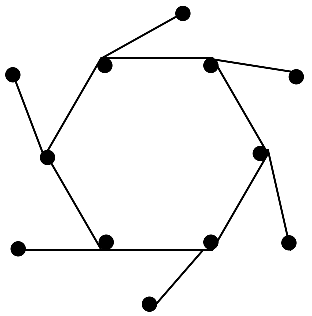
γ_{s2} (C_3^+) = 5
γ_{ns2} (C_3^+) = 4



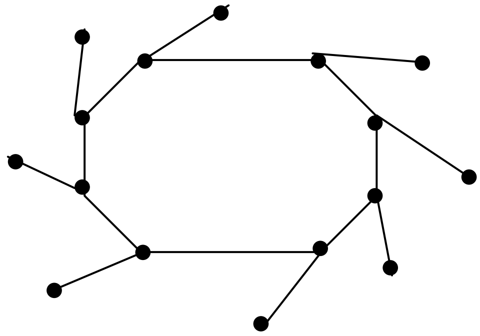
γ_{s2} (C_4^+) = 6
γ_{ns2} (C_4^+) = 6



γ_{s2} (C_5^+) = 7
γ_{ns2} (C_5^+) = 8

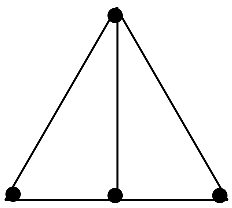


$\gamma_{s2}(C_6^+) = 8$
$\gamma_{ns2}(C_6^+) = 10$

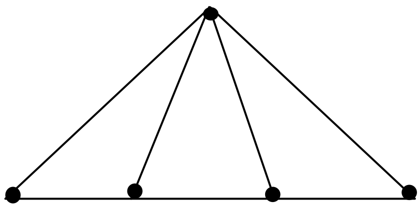


$\gamma_{s2}(C_8^+) = 12$
$\gamma_{ns2}(C_8^+) = 14$

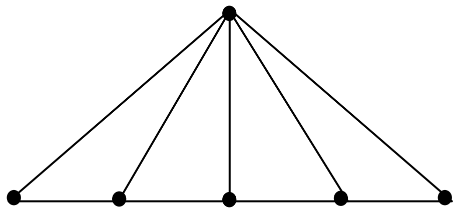
✓ For all fan graphs $\gamma_{ns2}(F_n) = n-1$, for $n \geq 4$ and
$\gamma_{s2}(F_n) = \lceil 2n/3 \rceil$, for $n \geq 4$



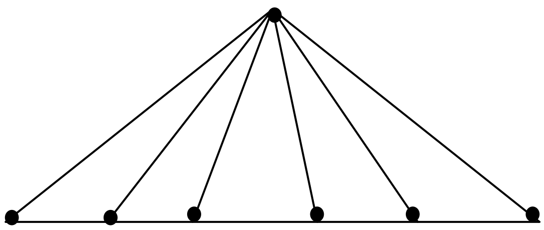
$\gamma_{s2}(F_3) = 2$
$\gamma_{ns2}(F_3) = 3$



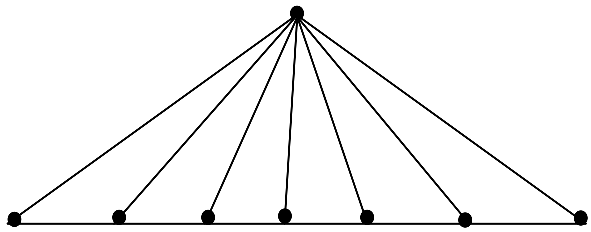
$\gamma_{s2}(F_4) = 3$
$\gamma_{ns2}(F_4) = 3$



$\gamma_{s2}(F_5) = 3$
$\gamma_{ns2}(F_5) = 4$

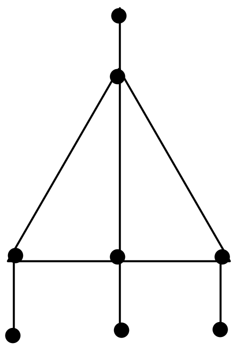


$\gamma_{s2}(F_6) = 4$
$\gamma_{ns2}(F_6) = 5$

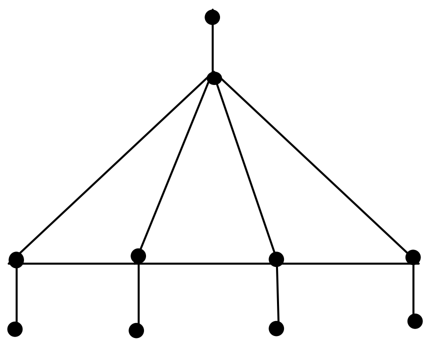


$\gamma_{s2}(F_7) = 4$
$\gamma_{ns2}(F_7) = 6$

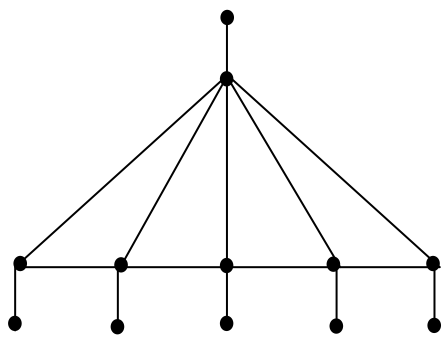
✓ $\gamma_{ns2}(F_n^+) = n+3$ for $4 \leq n \leq 6$
$n+4$ for $n > 6$ and
$\gamma_{s2}(F_n^+) = n+4$ for $4 \leq n \leq 6$
$n+5$ for $n > 6$



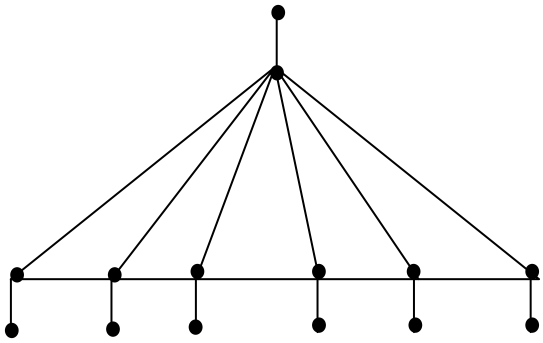
$\gamma_{s2}(F_3^+) = 6$
$\gamma_{ns2}(F_3^+) = 5$



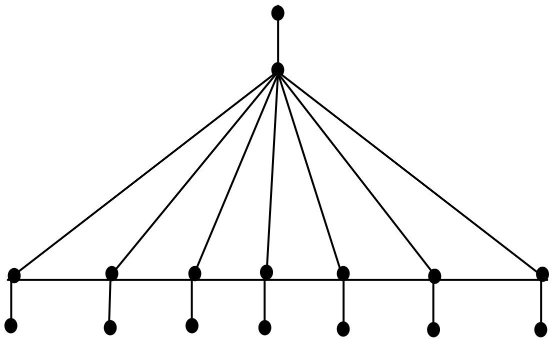
$\gamma_{s2}(F_4^+) = 8$
$\gamma_{ns2}(F_4^+) = 7$



$\gamma_{s2}(F_5^+) = 9$
$\gamma_{ns2}(F_5^+) = 8$

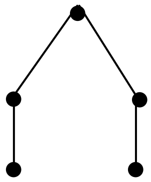


γ_{s2} $(F_6^+) = 10$
γ_{ns2} $(F_6^+) = 9$

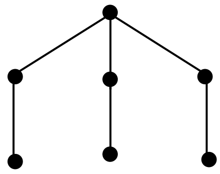


γ_{s2} $(F_7^+) = 12$
γ_{ns2} $(F_7^+) = 11$

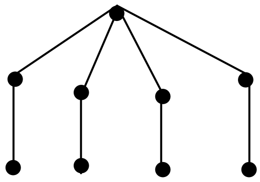
- ✓ $\gamma_{s2}(S(K_{1,n})) = n+1$ and $\gamma_{ns2}(S(K_{1,n})) = 2n$



γ_{s2} $(S(K_{1,2})) = 3$
γ_{ns2} $(S(K_{1,2})) = 3$

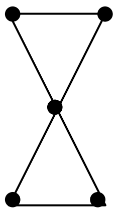


γ_{s2} $(S(K_{1,3})) = 4$
γ_{ns2} $(S(K_{1,3})) = 4$

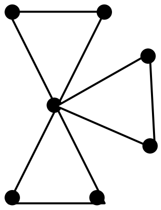


γ_{s2} $(S(K_{1,4})) = 5$
γ_{ns2} $(S(K_{1,4})) = 5$

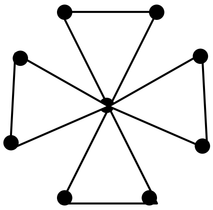
- ✓ For all γ_{ns2} $(C_n^{(t)}) = (n-1)t$ for $n \geq 3$ and
  γ_{s2} $(C_n^{(t)}) = t+1$ for n=3,
  $=(n-3)t+1$ for $n \geq 4$
  If n=3, it is friendship graph.



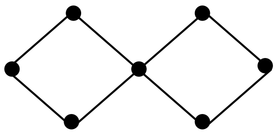
γ_{s2} $(C_3^{(2)}) = 3$
γ_{ns2} $(C_3^{(2)}) = 4$



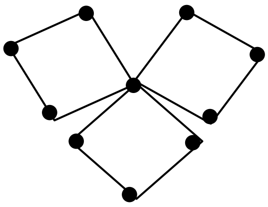
γ_{s2} $(C_3^{(3)}) = 4$
γ_{ns2} $(C_3^{(3)}) = 6$



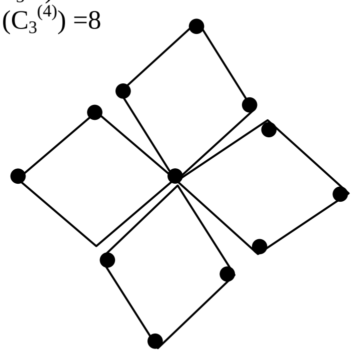
γ_{s2} $(C_3^{(4)}) = 5$
γ_{ns2} $(C_3^{(4)}) = 8$



γ_{s2} $(C_4^{(2)}) = 3$
γ_{ns2} $(C_4^{(2)}) = 6$



γ_{s2} $(C_4^{(3)}) = 4$
γ_{ns2} $(C_4^{(3)}) = 9$



γ_{s2} $(C_4^{(4)}) = 5$
γ_{ns2} $(C_4^{(4)}) = 12$

2. Main Results:

Observation: $\gamma_{s2}(G) \leq \gamma_{ns2}$ (G). C_4^+ is the graph for which $\gamma_{s2}(G) = \gamma_{ns2}$ (G) =4.

Theorem 2.1: For any graph G, γ (G) $\leq \gamma_{s2}$ (G), and γ(G) $\leq \gamma_{ns2}$ (G)

Proof: Since every Split two dominating set is a dominating set of G, γ (G) $\leq \gamma_{s2}$ (G), similarly, every Non Split two dominating set is a dominating set of G, γ (G) $\leq \gamma_{ns2}$ (G).

Observation: $\gamma(G) \leq \gamma_2(G) \leq \gamma_{s2}(G)$

Theorem 2.2: For any graph G, $\gamma(G) \leq \min\{\gamma_{s2}(G), \gamma_{ns2}(G)\}$

Proof: Since every Split two dominating set and every Non Split two dominating set of G are the dominating set of G, we have $\gamma(G) \leq \gamma_{s2}(G)$ and $\gamma(G) \leq \gamma_{ns2}(G)$ and hence $\gamma(G) \leq \min\{\gamma_{s2}(G), \gamma_{ns2}(G)\}$

Theorem 2.3: For any graph G, $\gamma_{s2}(G) \leq n - \delta(G)$, where $\delta(G)$ is the minimum degree among the vertices of G.

Note: For any tree T, $\delta(T) = 1$, hence $\gamma_{s2}(G) \leq n - 1$

Theorem 2.4: For any graph G, $\gamma_{ns2}(G) = n$ if and only if $\gamma_{s2}(G)$ contains either a pendant vertex of G or has two vertices of degree two or both.

Proof: Assume that $\gamma_{ns2}(G) = 0$, then there exists no non split two dominating set in G. If D is a split two dominating set, then the induced subgraph <V-D> is disconnected and every vertex in <V-D> is adjacent to atleast two vertices in D. As G is a tree, $|E(G)| < |V(G)|$, it is clear that D has a pendant vertex or it has two vertices of degree two or both. Conversely assume that Split two dominating set satisfies the sufficient condition. If D contains a pendant vertex, then it will not be present in the induced subgraph <V-D>. If D contains a two vertices u,v of degree two, then u,v will not belong to the induced subgraph <V-D>. Thus in both the cases, the induced subgraph <V-D> is disconnected. Therefore D is Split two dominating set with the sufficient condition. Hence $\gamma_{ns2}(G) = n$.

Corollary 2.5: For any tree t $\gamma_{ns2}(G) = n$

Proof: Every Split two dominating set of T contains a pendant edge of T; the result follows from the sufficient part of the previous theorem.

Note: If H is any spanning subgraph of G, then $\gamma(G) \leq \gamma(H)$.

Theorem 2.6: Let G be a graph which is not a cycle with atleast four vertices. If H is a connected spanning subgraph of G, then $\gamma_{s2}(G) \leq \gamma_{s2}(H)$.

Proof If G is connected, then any spanning tree T of G is minimally connected subgraph of G such that, $\gamma_{s2}(G) \leq \gamma_{s2}(T) \leq \gamma_{s2}(H)$. Hence the result.

Theorem 2.7: Let G be a graph which is not a cycle with atleast five vertices. If H is a connected spanning subgraph of G, then $\gamma_{ns2}(G) \leq \gamma_{ns2}(H)$.

Proof: The result follows from the previous theorem.

3. Bounds onγ_{s2} (G) and γ_{ns2} (G):

Theorem 3.1: If T is a tree which is not a star, then $\gamma_{ns2}(G) = n$ and $\gamma_{s2}(G) = n - 1$ for all $n \geq 3$.

Proof: Since T is not a star, there exists two adjacent cut vertices u and v with degree u and degree $v \geq 2$. This implies that $V - \{u,v\}$ is disconnected. Thus the theorem is true.

Characterization of Minimal Split and Non Split Two Dominating Set:

Theorem 3.3: A Split two dominating set is minimal if and only if for each vertices $u,v \in D$, one of the following condition is satisfied:

(i) There exists a vertex $w \in V-D$ such that $N_{s2}(w) \cap D = \{u,v\}$
(ii) v is not an isolated vertex in <D>
(iii) (<V-D> $\cup$ {u,v}) is connected.

Proof: Suppose D is a minimal split two dominating set such that {u,v} does not satisfy any of the above conditions. Then by (i) and (ii) D-{u,v} is a domination set, also since (iii) is not satisfied, <V-D> is disconnected. Therefore D-{u,v} is a Split two dominating set contradicting the minimal of D. Hence v satisfies one of the above conditions, and the bound is sharp.

Theorem 3.4: For all trees, $\gamma_{s2}(G) \leq (n\Delta(G)) / (\Delta(G)+1)$

Proof: Let D be a Split two dominating set of G, since D is minimal for every vertices $\{u,v\} \in D$,
(<V-D> $\cup$ {u,v}) is connected. Therefore $|V-D| \geq 2$, <V-D> is a two dominating set of G.
$\gamma_2(G) \leq |V-D| \leq n-\gamma_{s2}(G)$. But $\gamma_2(G) \geq \gamma(G) \geq n / \Delta+1$.Hence $\gamma_{s2}(G) \leq n-\gamma_2(G) \leq n- [n/\Delta+1] = (n\Delta+n-n)/(\Delta+1) = n\Delta/\Delta+1$. Hence $\gamma_{s2}(G) \leq n\Delta(G) / \Delta(G) +1$.

Theorem 3.5: For any tree G, $\gamma_{s2}(G) \leq q$, where q be the number of edges.

Proof: Any edge set containing q edges is clearly a Split two dominating set of G, which proves the result.

The Following Nordhaus – Gaddum Type Result is Immediate from the Above Theorem:

Theorem 3.6: Let G be a (p,q) tree graph such that G and its complement $\overline{G}$ are connected. Then

(i) $\gamma_{s2}(G) + \gamma_{s2}(\overline{G}) \leq ½ [p(p-1) – 4]$
(ii) $\gamma_{s2}(G) + \gamma_{s2}(\overline{G}) \leq ½ [pq(p-1) – 2 (q^2 + q)]$

Proof: Let q be the size of G, then the size of $\overline{G}$ is $[p(p-1)/2] – q – 1$. By the previous theorem
$\gamma_{s2}(G) \leq q$ and $\gamma_{s2}(\overline{G}) \leq [p(p-1)/2] – q – 1$. The results are immediate.

Theorem 3.7: If $\kappa(G) > \beta_0$, then $\gamma_{ns2}(G) = \gamma(G)$, where $\kappa(G)$ is the connectivity of G and $\beta_0(G)$ is the independence number of G.

Proof: Let D be a γ set of G, since $\kappa(G) > \beta_0(G) \geq \gamma(G)$, it implies that <V-D> is connected. This proves that D is a γ_{ns2} set of G. Hence $\gamma_{ns2}(G) = \gamma(G)$.

Theorem 3.8: For any graph G, γ_{ns2} (G) ≤ p- ω(G) + 1, where ω(G) is the clique number of G.

Proof: Let D be a set of vertices of G such that <D> is complete with |D| = ω(G). Then for any u∈D, (V-D) ∪ {u} is a nonsplit two dominating set of G. Thus γ_{ns2} (G) ≤ |V-D| +1 ≤ |V|-|D|+1≤ p-ω(G) +1.

Theorem 3.9: For any graph G, p- q/2 + q_0/2 ≤ γ_{s2} (G), where G is a (p,q) graph and q_0 = min q(<D>), where D∈ {minimal split two dominating sets of G}.

Proof: Let D be a minimal split two dominating set of G. Then for any point u∈V-D, there exist atleast two edges from u to D and there exists atleast 2|V-D| edges from V-D to D. Then the number of edges of G, q ≥ 2|V-D|+q_0=2|V|-2|D|+q_0 (or) 2|D| ≥ 2|V|-q+q_0⇒2|D| ≥ 2p-q+q_0⇒|D| ≥ p-q/2+q_0/2 which implies γ_{s2} (G) ≥ p-q/2+q_0/2.

Theorem 3.10: For any graph G, γ_{s2} (G) ≥ ⌈2p/Δ+2⌉

Proof: Every vertex in V-D contributes two to the degree sum of vertices of D, since 2|V-D| ≤ ∑d(u) for u∈D, where D is a split two dominating set. Therefore 2|V-D| ≤ ∑d(u) ≤ γ_{s2}. Δ, which implies 2|V|-2|D| ≤ γ_{s2}. Δ, which implies 2p-2γ_{s2} ≤ γ_{s2}. Δ , which implies γ_{s2}(Δ+2) ≥ 2p which implies γ_{s2} ≥ 2p/ (Δ+2). Hence γ_{s2} (G) ≥ ⌈2p/Δ+2⌉.

4. Conclusion:

The tools of number theory enable us to develop a simple method of constructing a graph with a given cardinality of the spilt dominating set with amazing ease. It is also amazing to observe how such a graph with a given domination number can be enlarged to include more vertices and edges in a methodical, simple manner without affecting the domination number. We can apply this to many applications such as to eradicate pests in Agriculture, to control viruses which produces diseases in an epidemic form, to maintain confidential in transferring the information, especially very useful for Defense sector. To some extent this may be due to the ever growing importance of computer science and its connection with graph theory.

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