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A STUDY ON MHD JEFFERY-HAMEL FLOW IN NANOFLUIDS USING NEW HOMOTOPY ANALYSIS METHOD

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Abstract:

In this paper, the magnetic effect of nanofluids on the non-liner Jeffery-Hamel flow is investigated. The basic running equations which are strongly non-liner are solved analytically by using the New Homotopy analysis method. The analytical expression of the dimensionless velocity profiles are derived with the help of nanoparticle solid volume fraction, Hartmann number and Reynolds number by using the New HAM. Our analytical results are compared with the previous work (DTM) and a satisfactory agreement is noted. This method can be easily extended to solve the other non-linear initial and boundary value problem is fluids flow problem.

Key Words: Jeffery-Hamel Flow, MHD, Nano Fluids, Differential Transformation Method & New Homotopy Analysis Method.

1. Introduction:

The study of an incompressible viscous fluid flow through convergent-divergent channels is one of the most applicable fields such as fluid mechanics, civil environment, bio-mechanical engineering. This flow situation was initially introduced by Jeffery et al [4] and Hamel et al[3]. After some years, this problem was elaborately researched by various researchers. The theory of MHD is the study of the interaction between magnetic fields and moving conducting fluids. The Jeffery Hamel problem deals with other methods such as the Homotopy perturbation method (HAM), the Adomain decomposition method (ADM) and the Spectral- Homotopy analysis method. Earlier, these three analytical methods applied to find approximation analytical solution of Jeffery Hamel flow were used by Joneidi et al[5]. The description of theoretical part of this problem consulting with magnetic hydrodynamic (MHD) flows of viscous fluids have been performed before ten years due to their rapidly increasing applications in many fields of technology and engineering. Such applications include MHD power generation MHD flows meters and MHD pumps by Sutton et al[6]. One type of MHD problems is Jeffery Hamel problem. The third order non-linear ordinary differential equation describes this Jefferyet al [4] and Hamel et al [3] have studied this problem before Jeffery-Hamel flow is an exact similarity solution of the Navier-stokes equation in the special case of two dimensional flow through a channel with inclined plane walls meeting at a vortex and with a source or sink at the vortex by Moghimiet al [1]. Recently, the effect of magnetic field and nanoparticle on the Jeffery-Hamel flow using a powerful analytical method called Adomain decomposition method was studied by Sheikholeslami et al [2]. The aim of this problem is using by New Homotopy method to upshot the approximation solutions of running equation with the effect of nanoparticle volume fraction on velocity profiles. Considering aspects, the combined effect of nanoparticles and magnetic field to the relation among velocity field in the Jeffery-Hamel flow problem using Channel angle and Reynolds number in the flow using modified HAM is not addressed yet.

2. Mathematical Formulation of the Problem:

In this section, we study the steady fully developed flow of an incompressible conducting viscous fluid between two rigid plane walls that meet at an angle 2α as shown in Fig. 1. The rigid walls are considered to be divergent if $\alpha > 0$ and convergent if $\alpha < 0$. We assume that the velocity is purely radial and depends on r and θ so that $v = (u(r,\theta),0)$ only and further there is no magnetic field in the z-direction. The following equation, the Navier -Stokes equation and Maxwell's equations in polar coordinates are given below

$$\frac{\rho_{nf}}{r}\frac{\partial(ru)}{\partial r} = 0 \qquad (1)$$

$$\frac{\mu_{nf}}{\rho_{nf}}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \theta^2} - \frac{u}{r^2}\right) - u\frac{\partial u}{\partial r} - \frac{1}{\rho_{nf}}\frac{\partial P}{\partial r} - \frac{\sigma B_0^2}{\rho_{nf} r^2}u = 0 \qquad (2)$$

$$\frac{1}{\rho_{nf}}\frac{\partial \rho}{\partial \theta} - \frac{2\mu_{nf}}{\rho_{nf} r^2}\frac{\partial u}{\partial \theta} = 0 \qquad (3)$$

Here $u = u(r,\theta)$ is the velocity, P is the pressure, B_0 is the electromagnetic induction, σ is the conductivity of the fluid, μ_{nf} is the effective viscosity, ρ_{nf} is the effective density and K_{nf} is the effective thermal conductivity of nanofluid.

The boundary conditions are

At the centerline of the channel: $\frac{\partial u}{\partial \theta} = 0$,

At the boundary of the channel: u=0.

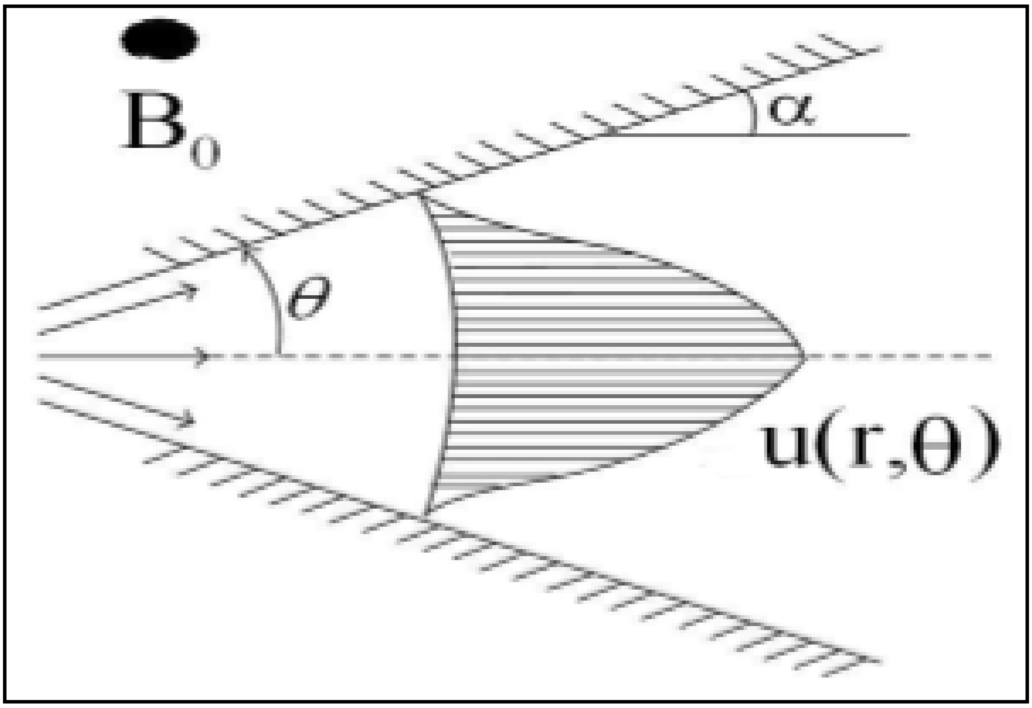

Fig.1 Geometry of the MHD Jeffery-Hamel flow in convergent/divergent channel with angle 2α

We follow this the paper, the thermo physical properties of the nanofluid as they are given in see the Table 1.Also the density of the nanofluid ρ_{nf} and the viscosity of the nanofluid μ_{nf} are given by the expressions

$$(\rho\beta)_{nf} = (1-\phi)(\rho\beta)_f + \phi(\rho\beta)_s,\ \mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}} \quad (4)$$

Here ϕ is the solid volume fraction, the subscripts nf, f and s respectively are the thermo-physical properties of the nanofluids, base fluid and the solid nanoparticles.
Considering only radial flow, the continuity Eqns. (1) follows that

$$u(r,\theta) = \frac{f(\theta)}{r} \quad (5)$$

Following [4] we define the dimensionless parameters

$$F(\eta) = \frac{f(\theta)}{f_{\max}} \text{ where } \eta = \frac{\theta}{\alpha} \quad (6)$$

Substituting the eqns. (4) - (6) into the eqns. (2) and(3) and eliminating the pressure term yields the nonlinear ordinary differential equation

$$F'''(\eta) + 2\alpha \operatorname{Re} A(1-\phi)^{2.5} F(\eta)F'(\eta) + (4-(1-\phi)^{2.5} M^2)\ \alpha^2 F'(\eta) = 0 \quad (7)$$

Here α denotes the angle between two plates, A is a parameter, Re denotes a Reynolds number and M denotes Hartmann numbers based on the electromagnetic parameter are introduced as follows:

$$A = 1-\phi+\frac{\rho_s}{\rho_f}\phi\ Re = \frac{f_{max}\rho_f\alpha}{\mu_f} = \frac{U_{max}\rho_f r\alpha}{\mu_f}, M^2 = \frac{\sigma B_0^2}{\mu_f}, \quad (8)$$

Where U_{max} is the velocity at the center of the channel $(r=0)$ and the Reynolds number.

$$Re = \begin{pmatrix} divergent - channel: & \alpha > 0, f_{max} > 0 \\ convergen - channel: & \alpha > 0, f_{max} > 0 \end{pmatrix}$$

According to the relation(5) and (6), the boundary conditions will be

$$F(0) = 1, F'(0) = 0 \quad (9)$$

$$F(1) = 0 \quad (10)$$

Obviously, these boundary conditions mean that maximum values of velocity are observed at centerline $\eta = 0$ as shown in Fig. 1. Thus, rate of velocity is zero at $\eta = 0$ .Also, the no-slip condition at a solid boundary is considered.

3. Analytical Expressions of Non-Linear Differential Eqns. (7), (9) And (10) Using the Modified Homotopy Analysis Method:

HAM is a non perturbative analytical method for obtaining series solutions to nonlinear equations and has been successfully applied to numerous problems in science and engineering [7-22]. In comparison with other perturbative and non perturbative analytical methods, HAM offers the ability to adjust and control the convergence of a solution via the so-called convergence-control parameter. Because of this, HAM has proved to be the most effective method for obtaining analytical solutions to highly non-linear differential equations. Previous applications of HAM have mainly focused on non-linear differential equations in which the non-linearity is a polynomial in terms of the unknown function and its derivatives. As seen in (1), the non-linearity present in electro hydrodynamic flow takes the form of a rational function, and thus, poses a greater challenge with respect to finding approximate solutions analytically. Our results show that even in this case, HAM yields excellent results.

Liao [7-15] proposed a powerful analytical method for non-linear problems, namely the Homotopy analysis method. This method provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution and to obtain numerical values from the infinite power series. In order to investigate the accuracy of the Homotopy analysis method (HAM) solution with a finite number of terms, the system of differential equations were solved. The Homotopy analysis method is a good technique comparing to another perturbation method. The Homotopy analysis method contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series. Using this method, we can obtain the following solution to (1) and (2) (see Appendix B).

$$F(y)=\frac{\cos\sqrt{k}-\cos\sqrt{k}y}{\cos\sqrt{k}-1}-h\left(\left(\frac{-k_1}{2(\cos\sqrt{k}-1)^2}\left[\frac{2\cos\sqrt{k}y}{k^2}-\frac{\cos 2ky}{8k^2}\right]\right)-\left(\frac{k_2}{\cos\sqrt{k}-1}\left[\frac{\cos ky}{k^2}\right]\right)+s\frac{y^2}{2}+u\right) \quad (11)$$

Where, $$k=2\alpha\operatorname{Re}A(1-\phi)^{2.5}+\left[\left(4-(1-\phi)^{2.5}\ M^2\right)\alpha^2\right] \quad (12)$$

$$k_1=2\alpha\operatorname{Re}A(1-\phi)^{2.5} \quad (13)$$

$$k_2=\left(\left[\left(4-(1-\phi)^{2.5}\ M^2\right)\alpha^2\right]\right) \quad (14)$$

$$s=2\left[\left(\frac{k_1}{2(\cos\sqrt{k}-1)^2}\left(\frac{2\cos\sqrt{k}\cos k}{k^2}-\frac{\cos 2k}{8k^2}\right)\right)+\left(\frac{k_2\cos k}{(\cos\sqrt{k}-1)k^2}\right)-\left(\frac{k_1}{2(\cos\sqrt{k}-1)^2}\left(\frac{2\cos\sqrt{k}}{k^2}-\frac{1}{8k^2}\right)\right)-\frac{k_2}{(\cos\sqrt{k}-1)}\left(\frac{1}{k^2}\right)\right] \quad (15)$$

$$u=\left[\left(\frac{k_1}{2(\cos\sqrt{k}-1)^2}\left(\frac{2\cos\sqrt{k}}{k^2}-\frac{1}{8k^2}\right)\right)-\frac{k_2}{(\cos\sqrt{k}-1)}\left(\frac{1}{k^2}\right)\right] \quad (16)$$

4. Results and Discussion:

In this section, Figure 1 shows that MHD Jeffery-Hamel flow in convergent/divergent channel with angle 2α .Figure (2)-(5) we discuss about the effect of the dimensionless velocity $(f(\eta))$ versus the dimensionless angle(η) with various value of parameters $\alpha, \operatorname{Re}, M$ andϕ.Figure 2 represents for some given fixed values of $\alpha, \operatorname{Re}, M$ andϕ.We observe that the fluidvelocity profiles decreases in divergent channel. Figure 3(a) for some given fixed values of $\alpha, \operatorname{Re}, M$ and varying values ofϕ. We infer that the solid volume fraction ϕ increases the fluid decreases. Figure 3 (b) represents for some given fixed values of $\alpha, \operatorname{Re}, M$ and various values ofϕ. We noted that the solid volume fraction ϕ increases the fluid decreases. That is, the fluid velocity is more for base fluid and less for nanofluid. Figure 4 represents for some given fixed values of $\alpha, \operatorname{Re}$and various values of$M, \phi$. It shows that the Hartmann number increases the fluid velocity increases for both viscous and nanofluid. Figure 5 represents the for some given fixed values of α, M and varying values ofϕ, Re. It evident that Reynolds number increases the fluid velocity decreases for both viscous and nanofluid

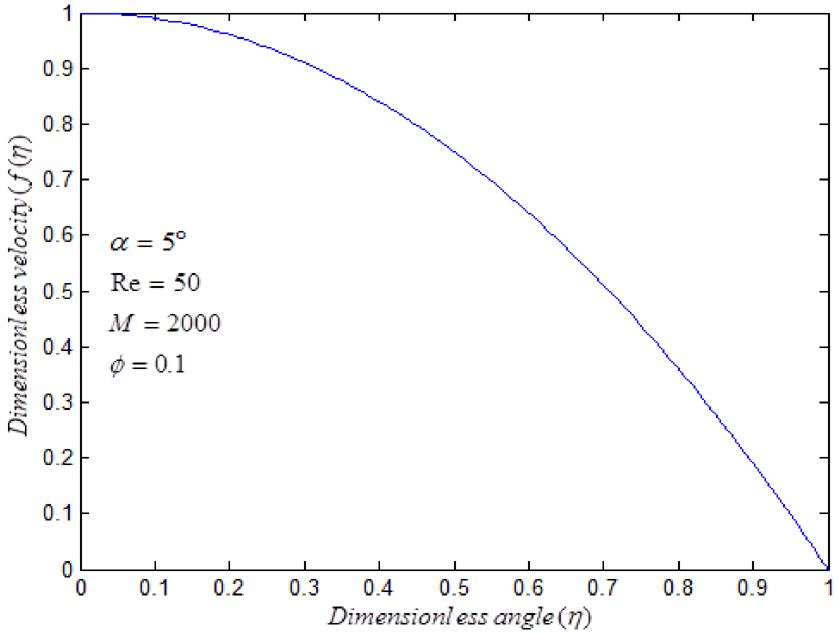

Fig.2: Dimensionless velocity $f(\eta)$ versus dimensionless angle (η). The curve is plotted for various values of the dimensionless parameters α, Re, M, ϕ using the eqn. (11) with *h=0.9.*

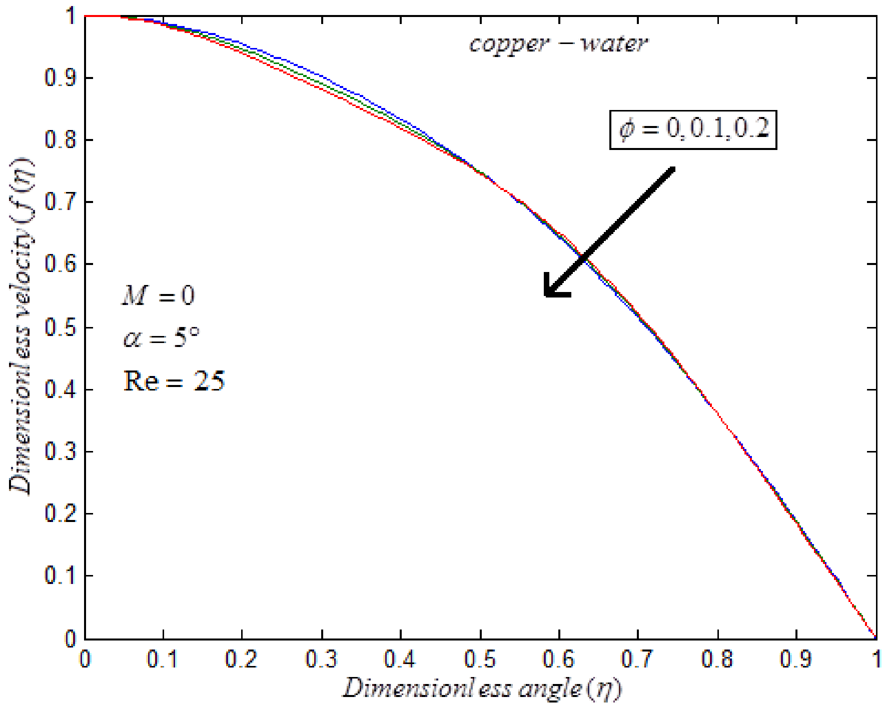

Fig.3(a): Dimensionless velocity $f(\eta)$ versus dimensionless angle (η). The curves are plotted for various values of solid volume fraction (ϕ) and in some fixed values of the other dimensionless parameters α, Re, M using the eqn. (11) with *h= 0.98*.

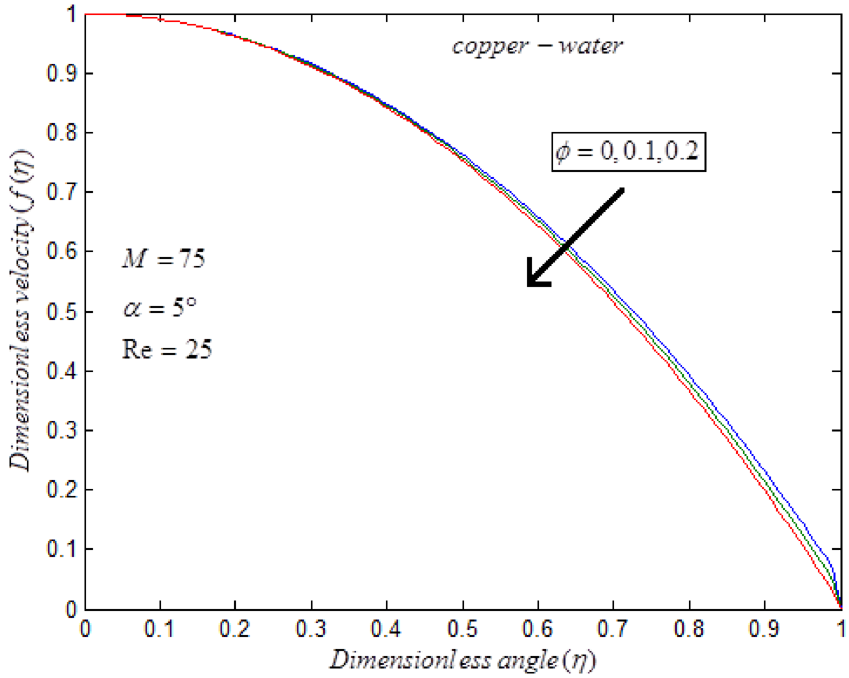

Fig.3(b): Dimensionless velocity $f(\eta)$ versus dimensionless angle (η). The curves are plotted for various values of solid volume fraction (ϕ) and in some fixed values of the other dimensionless parameters α, Re, M using the eqn. (11) with *h= 0.97*.

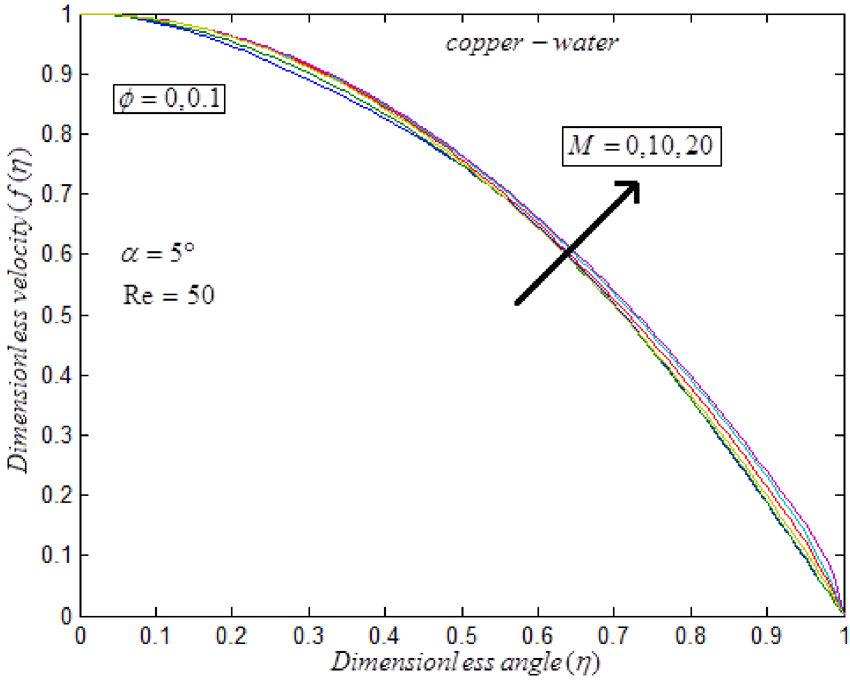

Fig.4: Dimensionless velocity $f(\eta)$ versus dimensionless angle (η). The curves are plotted for various values of solid volume fraction (ϕ), Hartmann number (M) and in some fixed values of the other dimensionless parameters α, Re using the eqn. (11) with *h= 0.97*.

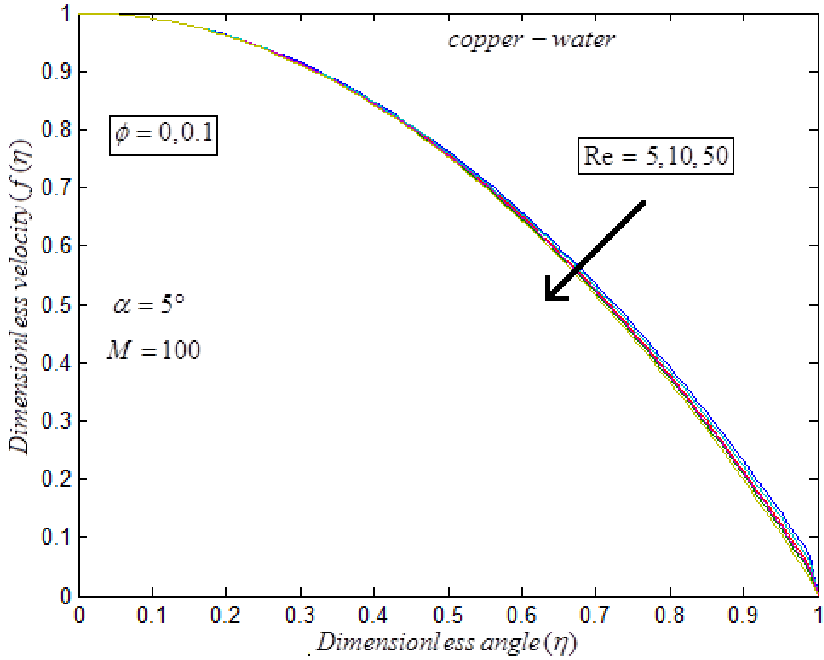

Fig.5: Dimensionless velocity $f(\eta)$ versus dimensionless angle (η). The curves are plotted for various values of solid volume fraction (ϕ), Reynolds number (Re) and in some fixed values of the other dimensionless parameters α, M using the eqn. (11) with *h=0.97*.

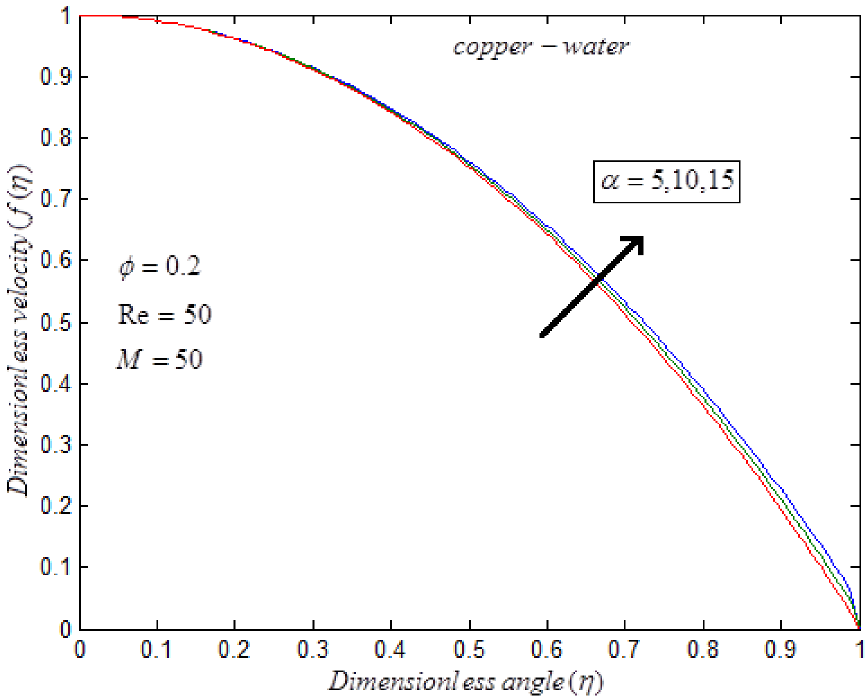

Fig.6: Dimensionless velocity $f(\eta)$ versus dimensionless angle (η). The curves are plotted for various values of channel angle (α) and in some fixed values of the other dimensionless parameters Re, M, ϕ using the eqn. (11) with *h=0.975*.

Table 1: Comparison between DTM – NHAM solution for velocity profiles when $\phi = 0,\ \mathrm{Re} = 50,\ \alpha = -5°,\ M = 50.$ and $h = 0.44878$

η	*NHAM*	*DTM*
0	1	1
0.1	0.996780	0.996756
0.2	0.986480	0.986491
0.3	0.967510	0.967516
0.4	0.936768	0.936738
0.5	0.889250	0.889208
0.6	0.817444	0.817461
0.7	0.710655	0.710623
0.8	0.553344	0.553388
0.9	0.325189	0.325142
1	0	0

Table 2: Comparison between DTM – NHAM solution for velocity profiles when $\phi = 0,\ \mathrm{Re} = 50,\ \alpha = 7.5°,\ M = 50.$ and $h = 0.6404$

η	*NHAM*	*DTM*
0	1	1
0.1	0.993799	0.993761
0.2	0.974719	0.974790
0.3	0.942219	0.942283
0.4	0.894719	0.894750
0.5	0.829749	0.829759
0.6	0.743499	0.7434936
0.7	0.630049	0.630013
0.8	0.479936	0.479970
0.9	0.278370	0.278323
1	0	0

Table 3: Comparison between DTM – NHAM solution for velocity when $\phi = 0.1,\ \mathrm{Re} = 5,\ \alpha = 5°,\ M = 100$ and $h = 0.93551$

η	*NHAM*	*DTM*
0	1	1
0.1	0.989400	0.989433
0.2	0.957800	0.957802
0.3	0.905320	0.905313
0.4	0.832320	0.832308
0.5	0.739250	0.739270
0.6	0.626860	0.626815
0.7	0.495692	0.495699
0.8	0.346880	0.346815
0.9	0.181171	0.181193
1	0	0

Table 4: Comparison between DTM – NHAM solution for velocity when $\phi = 0,\ \mathrm{Re} = 25,\ \alpha = 5°$ and $h = 0.8736$

$M = 0$		
η	*NHAM*	*DTM*

0	1	1
0.1	0.986616	0.986669
0.2	0.947224	0.947251
0.3	0.883412	0.883404
0.4	0.797645	0.797674
0.5	0.693288	0.693202
0.6	0.573346	0.573389
0.7	0.441577	0.441558
0.8	0.360171	0.300645
0.9	0.152987	0.152962
1	0	0
$M = 500$		
η	*NHAM*	*DTM*
0	1	1
0.1	0.990200	0.990221
0.2	0.960920	0.960939
0.3	0.912295	0.912287
0.4	0.844496	0.844405
0.5	0.757300	0.757317
0.6	0.650764	0.650758
0.7	0.523916	0.523953
0.8	0.375360	0.375331
0.9	0.202150	0.202153
1	0	0

Table 5: Comparison between DTM – NHAM solution for velocity when $\phi = 0.2,\ \mathrm{Re} = 25,\ \alpha = 5°$ and $h = 0.98442$

$M = 0$		
η	*NHAM*	*DTM*
0	1	1
0.1	0.984720	0.984743
0.2	0.939924	0.939943
0.3	0.868333	0.868370
0.4	0.774139	0.774186
0.5	0.662303	0.662374
0.6	0.538158	0.538121
0.7	0.406279	0.406256
0.8	0.360171	0.270828
0.9	0.134767	0.134828
1	0	0
$M = 200$		
η	*NHAM*	*DTM*
0	1	1
0.1	0.986139	0.986132
0.2	0.945279	0.945255
0.3	0.879408	0.879425
0.4	0.791791	0.791774

0.5	0.685999	0.686043
0.6	0.566091	0.566012
0.7	0.435127	0.435188
0.8	0.296639	0.296318
0.9	0.151119	0.151112
1	0	0

5. Conclusion:

We conclude that, using New Homotopy method, the analytical approximation solution of MHD on Jeffery-Hamel flow problem in the nanofluid could be determined. The approximate analytical expression of the dimensionless velocity has been derived by using the New Homotopy analysis method. The HAM solution has been compared with the DTM solution. The obtained result is reliable and it leads to many applications of non-linear ordinary differential equations. We also discussed the influence of various physical parameters on velocity in detail.

6. Acknowledgement:

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Appendix A

Basic Concept of HAM:

Consider the following differential equation:

$$N[u(t)] = 0 \qquad \text{(A.1)}$$

Where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao (2012) constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p) - u_0(t)] = phH(t)N[\varphi(t;p)] \qquad \text{(A.2)}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$, $\varphi(t:p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1$, it holds:

$$\varphi(t;0) = u_0(t) \text{ and } \varphi(t;1) = u(t) \tag{A.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) p^m \tag{A.4}$$

where

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \varphi(t;p)}{\partial p^m}\Big|_{p=0} \tag{A.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (A.4) converges at $p=1$ then we have:

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t). \tag{A.6}$$

Differentiating (A.2) for m times with respect to the embedding parameter p, and then setting $p = 0$ and finally dividing them by $m!$, we will have the so-called mth -order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\Re_m(\vec{u}_{m-1}) \tag{A.7}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}} \tag{A.8}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{A.9}$$

Applying L^{-1} on both side of eqn. (A7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \tag{A10}$$

In this way, it is easily to obtain u_m for $m \ge 1$, at M^{th} order, we have

5r

$$u(t) = \sum_{m=0}^{M} u_m(t) \tag{A.11}$$

When $M \to +\infty$, we get an accurate approximation of the original eqn.(A.1). For the convergence of the above method we refer the reader to Liao [20]. If an eqn. (A.1) admits unique solution, then this method will produce the unique solution.

Appendix B

Analytical Expressions of the Eqns. (7), (9) and (10) Using the New Modified Homotopy Analysis Method:

In this appendix, we derive the analytical expressions for $f(\eta)$ using the HAM.

The eqns. (7) can be written in the following form

$$F'''(\eta)+2\alpha\, Re\, A(1-\phi)^{2.5}F(\eta)F'(\eta)+(4-(1-\phi)^{2.5}M^2)\,\alpha^2F'(\eta)=0 \qquad \text{(B.1)}$$

We construct the homotopy for these equation as follows:

$$\begin{aligned}&(1-p)(F'''(\eta)+2\alpha\, Re\, A(1-\phi)^{2.5}F(0)F'(\eta)+(4-(1-\phi)^{2.5}M^2)\alpha^2F'(\eta))=\\&hp\,((F'''(\eta)+2\alpha\, Re\, A(1-\phi)^{2.5}F(\eta)F'(\eta)+(4-(1-\phi)^{2.5}M^2)\alpha^2F'(\eta))\end{aligned} \qquad \text{(B.2)}$$

The approximate solution of the eqns. (B.2)is given by

$$\eta=\eta_0+p\eta_1+p^2\eta_2+\ldots \qquad \text{(B.3)}$$

The initial approximations are as follows:

$$F_0(0)=1,\ F_0'(0)=0,\ F_0(1)=0 \qquad \text{(B.4 (a))}$$

$$F_i(0)=0,\ F_i'(0)=0,\ F_i(1)=0, \quad i=1,2,3,\ldots\ldots \qquad \text{(B. 4(b))}$$

Substituting the eqn.(B.3) into an eqn. (B.2) we get

$$\begin{aligned}&(1-p)(F'''(\eta_0+p\eta_1+\ldots)+2\alpha\, Re\, A(1-\phi)^{2.5}F(0)F'(\eta_0+p\eta_1+\ldots)\\&+(4-(1-\phi)^{2.5}M^2)\alpha^2F'(\eta_0+p\eta_1+\ldots))=\ hp((F'''(\eta_0+p\eta_1+\ldots)\\&+2\alpha\, Re\, A(1-\phi)^{2.5}F(\eta_0+p\eta_1+\ldots)(F'(\eta_0+p\eta_1+\ldots)\\&+(4-(1-\phi)^{2.5}M^2)\alpha^2F'(\eta_0+p\eta_1+\ldots))\end{aligned} \qquad \text{(B.5)}$$

Comparing the coefficients of like powers of p in eqn.(B.5), we get the following eqn.

$$p^0: F'''(\eta_0)+2\alpha\, Re\, A(1-\phi)^{2.5}F'(\eta_0)+(4-(1-\phi)^{2.5}M^2)\alpha^2F'(\eta_0)=0 \qquad \text{(B.6)}$$

$$\begin{aligned}&p^1: F'''(\eta_1)+2\alpha\, Re\, A(1-\phi)^{2.5}F'(\eta_1)+(4-(1-\phi)^{2.5}M^2)\alpha^2F'(\eta_1)\\&-h(F'''(\eta_0)-2\alpha h\, Re\, A(1-\phi)^{2.5}F(\eta)F'(\eta_0)-h(4-(1-\phi)^{2.5}M^2)\alpha^2F'(\eta_0))=0\end{aligned} \qquad \text{(B.7)}$$

Solving the eqns. (B.6), (B.7)and using boundary conditions (B.4), we obtained the following results:

$$F_0(y)=\frac{\cos\sqrt{k}-\cos\sqrt{k}\,y}{\cos\sqrt{k}-1} \qquad \text{(B.8)}$$

$$F_1(y)=\left(\frac{-k_1}{2(\cos\sqrt{k}-1)^2}\left[\frac{2\cos\sqrt{k}\,y}{k^2}-\frac{\cos 2ky}{8k^2}\right]\right)-\left(\frac{k_2}{\cos\sqrt{k}-1}\left[\frac{\cos ky}{k^2}\right]\right)+s\frac{y^2}{2}+u \qquad \text{(B.9)}$$

where,

k, k_1, k_2, s *and* u defined in the text eqns. (12)-(16) respectively.

According to the Homotopy analysis method we have

$$F=\lim_{p\to 1}F(y)=F_0+F_1 \qquad \text{(B.10)}$$

Using the eqns. (B.8)and (B.9) into an eqn. (B.10), we obtain the solution in the text eqn. (11).

Appendix C:

Table 1: Thermo- physical properties for pure water and copper nanoparticle.

Property	Pure water	Copper (cu)
$\rho(kg/m^2)$	997.1	8933
$\mu(Nm/s)$	1×10^{-3}	-
$\kappa(W/mK)$	0.613	400
$\beta(1/K)$	207×10^{-6}	17×10^{-6}

Appendix D:
Nomenclature:

Symbol	Meaning
η	Dimensionless angle
ϕ	Solid volume fraction
α	Channel angle
Re	Reynolds number
M	Hartmann number
$f(\eta)$	Dimensionless velocity
P	Pressure
B_0	Electromagnetic induction
σ	Conductivity
μ_{nf}	Effective viscosity
ρ_{nf}	Effective density
K_{nf}	Effective thermal conductivity