International Journal of Scientific Research and Modern Education (IJSRME)
Impact Factor: 7.137, ISSN (Online): 2455 – 5630
(www.rdmodernresearch.com) Volume 5, Issue 2, 2020

NUMERICAL SOLUTION OF INTUITIONISTIC FUZZY DIFFERENTIAL EQUATION USING RUNGE-KUTTA METHOD

Dr. M. Mary Jansi Rani* & K. Jamshida**
* Assistant Professor, Department of Mathematics, Holy Cross College, Trichirapalli, Tamilnadu
** Ph.D Research Scholar, Department of Mathematics, Holy Cross College, Trichirapalli, Tamilnadu

Cite This Article: Dr. M. Mary Jansi Rani & K. Jamshida, "Numerical Solution of Intuitionistic Fuzzy Differential Equation Using Runge-Kutta Method", International Journal of Scientific Research and Modern Education, Volume 5, Issue 2, Page Number 1-10, 2020.

Abstract:

The aim of this paper is to propose Runge-Kutta (ERK) methods of order 3 and 4 for solving intuitionistic fuzzy differential equations (IFDEs). Convergence analysis of RK methods has been carried out. The applicability of RK methods is illustrated by solving intuitionistic fuzzy differential equations with triangular intuitionistic fuzzy numbers. Comparison of the numerical solution with exact solution shows good accuracy.

Key Words: Intuitionistic Fuzzy Differential Equations, IVP, RK Methods, Convergence.

1. Introduction:

Fuzzy set theory is a useful tool to describe the situation in which data are imprecise or vague or uncertain. The concept of fuzzy set theory was introduced by Zadeh [20]. This concept was extended to the intuitionistic fuzzy set (IFS) theory by Atanassov [5]. Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of science and engineering. Chang and Zadeh [8] first introduced the concept of fuzzy derivative. Kaleva [9, 10] and Seikkala [17] extensively studied the fuzzy initial value problems. Recently many research articles are focused on numerical solution of fuzzy initial value problems. Ming Ma et al. [12] introduced Euler method for solving FDEs numerically, Ahmad and Hasan [3] have discussed a new fuzzy version of Euler's method with fuzzy initial values. Bede et al. [6, 7] introduced a new concept of the generalized of fuzzy interval valued functions. Melliani et al. [11] have discussed differential and partial differential equations under intuitionistic fuzzy environment. Abbasbandy and Allahviranloo [1, 2, 4] introduced the numerical solution of FDE by Runge-Kutta method with intuitionistic treatment. Mondal and Roy [19] havediscussed the first order homogeneous ordinary differential equation with initial value as triangular intuitionistic fuzzy number. Sneh Lata and Amit Kumar [18] have introduced time dependent intuitionistic fuzzy linear differential equation. Nirmala et al. [13 – 16] have discussed numerical solution of intuitionistic fuzzy differential equation by Euler, Modified Euler, fourth order RK and predictor-corrector methods under generalized differentiability concept. This paper presents RK methods of order 3 and 4 for solving intuitionistic fuzzy IVPs. The convergence analysis of RK methods has been discussed. The efficiency of these methods has been illustrated by numerical examples.

2. Preliminaries:

Definition 1:

The α , β -cut of an IFN $A=\{x,\mu_A(x),\vartheta_A(x)\setminus x\in X\}$ is defined as follows:

$$A=\{x,\mu_A(x),\vartheta_A(x)\setminus x\in X,\mu_A(x)\geq\alpha\, and\, \vartheta_A(x)\leq 1-\alpha\}\ \forall x\in[0,1]$$

The α-cut representation of IFN A generates the following pair of intervals and is denoted by $[\mathrm{A}]_\alpha=\{[\mathrm{A}_L^+(\alpha),\mathrm{A}_U^+(\alpha)],[\mathrm{A}_L^-(\beta),\mathrm{A}_U^-(\beta)]\}$.

Definition 2

A Triangular Intuitionistic Fuzzy Number (TIFN) U is an intuitionistic fuzzy set in R with the following membership function $\mu_A(x)$ and non-membership function $\vartheta_A(x)$. given as follows:

$$\mu_A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1}, & a_1\leq x\leq a_2\\ \dfrac{a_3-x}{a_3-a_2}, & a_2\leq x\leq a_3\\ 0, & \text{otherwise}\end{cases}\qquad \vartheta_A(x)=\begin{cases}\dfrac{a_2-x}{a_2-a_1'}, & a_1'\leq x\leq a_2\\ \dfrac{x-a_2}{a_3'-a_2}, & a_2\leq x\leq a_3'\\ 1, & \text{otherwise}\end{cases}$$

where $a_1{}' \le a_1 \le a_2 \le a_3 \le a_3{}'$ and TIFN is denoted by $U = (a_1, a_2, a_3;\ a_1{}',\ a_2,\ a_3{}')$

3. Intuitionistic Fuzzy Cauchy Problem:

Consider IFDE equation is of the form

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [\mathrm{a,b}] \\ y(t_0) = y_0 \end{cases} \tag{3.1}$$

Where y is an intuitionistic fuzzy variable $t,\ f(t, y(t))$ is an intuitionistic fuzzy function of the crisp variable t and the intuitionistic fuzzy variable y and y' is the intuitionistic fuzzy derivative. If an initial value $y(t_0) = y_0$. It is given by $y'(t) = f(t, y(t)),\ \ y(t_0) = y_0$

An equivalent system equations as follows:

$$y'(t) = \left\{ \underline{y}'^{+}(\mathrm{t};\alpha), \overline{y}'^{+}(\mathrm{t};\alpha)], [\underline{y}'^{-}(\mathrm{t};\alpha), \overline{y}^{-}(\mathrm{t};\alpha)] \right\}$$, where

$$\underline{y}'^{+}(\mathrm{t};\alpha) = \underline{f}^{+}(t, y^{+}) = \min\left\{ f^{+}(t.u) \,|\, u \in [\underline{y}^{+}, \overline{y}^{+}] \right\} = F(t,\ \underline{y}^{+}, \overline{y}^{+}),\quad \underline{y}^{+}(t_0) = \underline{y_0}^{+} \tag{3.2}$$

$$\overline{y}'^{+}(\mathrm{t};\alpha) = \overline{f}^{+}(t, y^{+}) = \max\left\{ f^{+}(t.u) \,|\, u \in [\underline{y}^{+}, \overline{y}^{+}] \right\} = G(t,\ \underline{y}^{+}, \overline{y}^{+}),\quad \overline{y}^{+}(t_0) = \overline{y_0}^{+} \tag{3.3}$$

$$\underline{y}'^{-}(\mathrm{t};\beta) = \underline{f}^{-}(t, y^{-}) = \min\left\{ f^{-}(t.u) \,|\, u \in [\underline{y}^{-}, \overline{y}^{-}] \right\} = H(t,\ \underline{y}^{-}, \overline{y}^{-}),\quad \underline{y}^{-}(t_0) = \underline{y_0}^{-} \tag{3.4}$$

$$\overline{y}'^{-}(\mathrm{t};\beta) = \overline{f}^{-}(t, y^{-}) = \max\left\{ f^{-}(t.u) \,|\, u \in [\underline{y}^{-}, \overline{y}^{-}] \right\} = I(t,\ \underline{y}^{-},\ \overline{y}^{-}),\quad \overline{y}^{-}(t_0) = \overline{y_0}^{-} \tag{3.5}$$

4. Runge-Kutta Methods for Solving Intuitionistic Fuzzy Differential Equations:

Consider the fuzzy IVP

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{4.1}$$

Where y is a fuzzy function of t, $f(t, y)$ is a fuzzy operation of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a fuzzy number.

Let $Y = [\underline{Y}^{+},\ \overline{Y}^{+}, \underline{Y}^{-},\ \overline{Y}^{-}]$ be the exact solution and $y = [\underline{y}^{+},\ \overline{y}^{+}, \underline{y}^{-},\ \overline{y}^{-}]$ be the approximated solution of the fuzzy IVP (4.1). Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$$[Y(t_n)]_{\alpha,\beta} = [\underline{Y}^{+}(t_n,\alpha), \overline{Y}^{+}(t_n,\alpha), \underline{Y}^{-}(t_n,\beta), \overline{Y}^{-}(t_n,\beta)]$$

$$[\mathrm{y}(t_n)]_{\alpha,\beta} = [\underline{y}^{+}(t_n,\alpha), \overline{y}^{+}(t_n,\alpha), \underline{y}^{-}(t_n,\beta), \overline{y}^{-}(t_n,\beta)],\ (0 \le \alpha, \beta \le N).$$

The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N},\ t_i = t_0 + ih,\ 0 \le i \le N. \tag{4.2}$$

The general form of s-stage RK for intuitionistic fuzzy IVPs

$$\begin{aligned} \underline{y}^{+}{}_{n+1} &= \underline{y}^{+}{}_{n} + h\sum_{i=1}^{s} b_i K_i \\ \overline{y}^{+}{}_{n+1} &= \overline{y}^{+}{}_{n} + h\sum_{i=1}^{s} b_i L_i \\ \underline{y}^{-}{}_{n+1} &= \underline{y}^{-}{}_{n} + h\sum_{i=1}^{s} b_i M_i \\ \overline{y}^{-}{}_{n+1} &= \overline{y}^{-}{}_{n} + h\sum_{i=1}^{s} b_i N_i \end{aligned} \tag{4.3a}$$

Where

$$K_i = F(t_n + c_i h,\ \underline{y}^{+}_{n} + h\sum_{j=1}^{s} a_{ij} K_j),$$

$$L_i = G(t_n + c_i h,\ \overline{y}^{+}_{n} + h\sum_{j=1}^{s} a_{ij} L_j),$$

$$M_i = H(t_n + c_i h,\ \underline{y}^{-}_{n} + h\sum_{j=1}^{s} a_{ij} M_j), \qquad (4.3b)$$

$$N_i = I(t_n + c_i h,\ \overline{y}^{-}_{n} + h\sum_{j=1}^{s} a_{ij} N_j), \quad i = 1, 2, \cdots, s$$

where a_{ij}'s, b_i's and c_i's are constants chosen according to the order of the RK methods. Here, the RK schemes up to fourth order have been provided.

4.1 First Order RK (RK1) for Intuitionistic Fuzzy IVP:

RK1 scheme for intuitionistic fuzzy IVP is given by

$$\underline{y}^{+}(t_{n+1};\ \alpha) = \underline{y}^{+}(t_n;\alpha) + hK_1$$
$$\overline{y}^{+}(t_{n+1};\ \alpha) = \overline{y}^{+}(t_n;\alpha) + hL_1$$
$$\underline{y}^{-}(t_{n+1};\ \alpha) = \underline{y}^{-}(t_n;\beta) + hM_1$$
$$\overline{y}^{-}(t_{n+1};\ \alpha) = \overline{y}^{-}(t_n;\beta) + hN_1$$

Where

$$K_1 = F(t_n, \underline{y}^{+}(t_n;\alpha), \overline{y}^{+}(t_n;\alpha))$$
$$L_1 = G(t_n, \underline{y}^{+}(t_n;\alpha), \overline{y}^{+}(t_n;\alpha))$$
$$M_1 = F(t_n, \underline{y}^{-}(t_n;\beta), \overline{y}^{-}(t_n;\beta))$$
$$N_1 = G(t_n, \underline{y}^{-}(t_n;\beta), \overline{y}^{-}(t_n;\beta))$$

This formula is called intuitionistic fuzzy as Euler method.

4.2 Second Order RK (RK2) for intuitionistic Fuzzy IVP:

RK2 scheme for intuitionistic fuzzy IVP is given by

$$\underline{y}^{+}(t_{n+1};\ \alpha) = \underline{y}^{+}(t_n;\alpha) + hK_2$$
$$\overline{y}^{+}(t_{n+1};\ \alpha) = \overline{y}^{+}(t_n;\alpha) + hL_2$$
$$\underline{y}^{-}(t_{n+1};\ \beta) = \underline{y}^{+}(t_n;\beta) + hM_2$$
$$\overline{y}^{-}(t_{n+1};\ \beta) = \overline{y}^{+}(t_n;\beta) + hN_2$$

Where

$$K_1 = F(t_n, \underline{y}^{+}(t_n;\alpha), \overline{y}^{+}(t_n;\alpha))$$
$$L_1 = G(t_n, \underline{y}^{+}(t_n;\alpha), \overline{y}^{+}(t_n;\alpha))$$
$$M_1 = H(t_n, \underline{y}^{-}(t_n;\beta), \overline{y}^{-}(t_n;\beta))$$
$$N_1 = I(t_n, \underline{y}^{-}(t_n;\beta), \overline{y}^{-}(t_n;\beta))$$

$$K_2 = F(t_n + \frac{1}{2}\mathrm{h},\ \underline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hK}_1, \overline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hL}_1)$$

$$L_2 = G(t_n + \frac{1}{2}\mathrm{h}, \underline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hK}_1, \overline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hL}_1)$$

$$M_2 = H(t_n + \frac{1}{2}\mathrm{h},\ \underline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hM}_1, \overline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hN}_1)$$

$$N_2 = I(t_n + \frac{1}{2}\mathrm{h}, \underline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hM}_1, \overline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hN}_1)$$

4.3 Third Order RK (RK3) for Intuitionistic Fuzzy IVP:

RK3 scheme for intuitionistic fuzzy IVP is given by

$$\underline{y^+}(t_{n+1};\ \alpha) = \underline{y^+}(t_n;\ \alpha) + \frac{h}{6}(\mathrm{K}_1 + 4K_2 + K_3)$$

$$\overline{y^+}(t_{n+1};\ \alpha) = \overline{y^+}(t_n;\ \alpha) + \frac{h}{6}(\mathrm{L}_1 + 4L_2 + L_3)$$

$$\underline{y^-}(t_{n+1};\ \beta) = \underline{y^-}(t_n;\ \beta) + \frac{h}{6}(\mathrm{M}_1 + 4M_2 + M_3)$$

$$\overline{y^-}(t_{n+1};\ \beta) = \overline{y^-}(t_n;\ \beta) + \frac{h}{6}(\mathrm{N}_1 + 4N_2 + N_3)$$

Where

$$K_1 = F(t_n, \underline{y^+}(\mathrm{t}_n;\alpha), \overline{y^+}(\mathrm{t}_n;\alpha))$$

$$L_1 = G(t_n, \underline{y^+}(\mathrm{t}_n;\alpha), \overline{y^+}(\mathrm{t}_n;\alpha))$$

$$M_1 = H(t_n, \underline{y^-}(\mathrm{t}_n;\beta), \overline{y^-}(\mathrm{t}_n;\beta))$$

$$N_1 = I(t_n, \underline{y^-}(\mathrm{t}_n;\beta), \overline{y^-}(\mathrm{t}_n;\beta))$$

$$K_2 = F(t_n + \frac{1}{2}\mathrm{h},\ \underline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hK}_1, \overline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hL}_1)$$

$$L_2 = G(t_n + \frac{1}{2}\mathrm{h}, \underline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hK}_1, \overline{y^+}(\mathrm{t}_n;\alpha) + \frac{1}{2}\mathrm{hL}_1)$$

$$M_2 = H(t_n + \frac{1}{2}\mathrm{h},\ \underline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hM}_1, \overline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hN}_1)$$

$$N_2 = I(t_n + \frac{1}{2}\mathrm{h}, \underline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hM}_1, \overline{y^-}(\mathrm{t}_n;\beta) + \frac{1}{2}\mathrm{hN}_1)$$

$$K_3 = F(t_n + \mathrm{h}, \underline{y^+}(\mathrm{t}_n;\alpha) + \mathrm{h}(-K_1 + 2K_2),\ \overline{y^+}(\mathrm{t}_n;\alpha) + h(-L_1 + 2L_2))$$

$$L_3 = G(t_n + \mathrm{h}, \underline{y^+}(\mathrm{t}_n;\alpha) + \mathrm{h}(-K_1 + 2K_2), \overline{y^+}(\mathrm{t}_n;\alpha) + \mathrm{h}(-L_1 + 2L_2))$$

$$M_3 = H(t_n + \mathrm{h}, \underline{y^-}(\mathrm{t}_n;\beta) + \mathrm{h}(-M_1 + 2M_2),\ \overline{y^-}(\mathrm{t}_n;\alpha) + h(-N_1 + 2N_2))$$

$$N_3 = I(t_n + \mathrm{h}, \underline{y^-}(\mathrm{t}_n;\beta) + \mathrm{h}(-M_1 + 2M_2), \overline{y^-}(\mathrm{t}_n;\alpha) + \mathrm{h}(-N_1 + 2N_2))$$

4.4 Fourth Order RK (RK4) for Intuitionistic Fuzzy IVP:

RK4 scheme for intuitionistic fuzzy IVP is given by

$$\underline{y}^{+}(t_{n+1};\ \alpha)=\underline{y}^{+}(t_n;\ \alpha)+\frac{h}{6}(\mathrm{K}_1+2K_2+2K_3+K_4)$$

$$\overline{y}^{+}(t_{n+1};\ \alpha)=\overline{y}^{+}(t_n;\ \alpha)+\frac{h}{6}(\mathrm{L}_1+2L_2+2L_3+L_4)$$

$$\underline{y}^{-}(t_{n+1};\ \beta)=\underline{y}^{-}(t_n;\ \beta)+\frac{h}{6}(\mathrm{M}_1+2M_2+2M_3+M_4)$$

$$\overline{y}^{+}(t_{n+1};\ \beta)=\overline{y}^{-}(t_n;\ \beta)+\frac{h}{6}(\mathrm{N}_1+2N_2+2N_3+N_4)$$

Where

$$K_1=F(t_n,\underline{y}^{+}(\mathrm{t}_n;\alpha),\overline{y}^{+}(\mathrm{t}_n;\alpha))$$

$$L_1=G(t_n,\underline{y}^{+}(\mathrm{t}_n;\alpha),\overline{y}^{+}(\mathrm{t}_n;\alpha))$$

$$M_1=H(t_n,\underline{y}^{-}(\mathrm{t}_n;\beta),\overline{y}^{-}(\mathrm{t}_n;\beta))$$

$$N_1=I(t_n,\underline{y}^{-}(\mathrm{t}_n;\beta),\overline{y}^{-}(\mathrm{t}_n;\beta))$$

$$K_2=F(t_n+\frac{1}{2}\mathrm{h},\ \underline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hK}_1,\overline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hL}_1)$$

$$L_2=G(t_n+\frac{1}{2}\mathrm{h},\underline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hK}_1,\overline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hL}_1)$$

$$M_2=H(t_n+\frac{1}{2}\mathrm{h},\ \underline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hM}_1,\overline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hN}_1)$$

$$N_2=I(t_n+\frac{1}{2}\mathrm{h},\underline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hM}_1,\overline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hN}_1)$$

$$K_3=F(t_n+\frac{1}{2}\mathrm{h},\ \underline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hK}_2,\overline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hL}_2)$$

$$L_3=G(t_n+\frac{1}{2}\mathrm{h},\underline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hK}_2,\overline{y}^{+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hL}_2)$$

$$M_3=H(t_n+\frac{1}{2}\mathrm{h},\ \underline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hM}_2,\overline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hN}_2)$$

$$N_3=I(t_n+\frac{1}{2}\mathrm{h},\underline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hM}_2,\overline{y}^{-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hN}_2)$$

$$K_4=F(t_n+\mathrm{h},\ \ \underline{y}^{+}(\mathrm{t}_n;\alpha)+hK_3,\ \overline{y}^{+}(\mathrm{t}_n;\alpha)+hL_3)$$

$$L_4=G(t_n+\mathrm{h},\ \ \underline{y}^{+}(\mathrm{t}_n;\alpha)+hK_3,\ \overline{y}^{+}(\mathrm{t}_n;\alpha)+hL_3)$$

$$M_4=H(t_n+\mathrm{h},\ \ \underline{y}^{-}(\mathrm{t}_n;\beta)+hK_3,\ \overline{y}^{-}(\mathrm{t}_n;\beta)+hL_3)$$

$$N_4=I(t_n+\mathrm{h},\ \ \underline{y}^{-}(\mathrm{t}_n;\beta)+hK_3,\ \overline{y}^{-}(\mathrm{t}_n;\beta)+hL_3)$$

5. Convergence of Intuitionistic Fuzzy Runge-Kutta Methods:

The solution is obtained by grid points at

$$a=t_0\le t_1\le\cdots\cdots\le t_N=b \text{ and } h=\frac{b-a}{N}=t_{n+1}-t_n \qquad (5.1)$$

We define

$$F[t_n,y(t_n;a)]=\sum_{i=1}^{s}b_iK_i(t_n,y(t_n;a))\quad G[t_n,y(t_n;a)]=\sum_{i=1}^{s}b_iL_i(,y(t_n;a))$$

$$H[t_n,y(t_n;\beta)]=\sum_{i=1}^{s}b_iM_i(t_n,y(t_n;\beta))\ I[t_n,y(t_n;\beta)]=\sum_{i=1}^{s}b_iN_i(t_n,y(t_n;\beta)) \qquad (5.2)$$

The exact and approximate solutions at $t_n,\ 0 \le n \le N$ are denoted respectively by

$$[\mathrm{Y}(\mathrm{t}_n)]_{\alpha,\beta} = [\underline{Y}^{+}(\mathrm{t}_n;\alpha), \overline{Y}^{+}(\mathrm{t}_n;\alpha), \underline{Y}^{-}(\mathrm{t}_n;\beta), \overline{Y}^{-}(\mathrm{t}_n;\beta)] \text{ and}$$

$$[\mathrm{y}(\mathrm{t}_n)]_{\alpha,\beta} = [\underline{y}^{+}(\mathrm{t}_n;\alpha), \overline{y}^{+}(\mathrm{t}_n;\alpha), \underline{y}^{-}(\mathrm{t}_n;\beta), \overline{y}^{-}(\mathrm{t}_n;\beta)].$$

We have

$$\underline{Y}^{+}(t_{n+1};\ \alpha) \approx \underline{Y}^{+}(t_n;\ \alpha) + h\,\mathrm{F}[\mathrm{t}_n, \underline{Y}^{+}(\mathrm{t}_n;\alpha), \overline{Y}^{+}(\mathrm{t}_n;\alpha)],$$

$$\overline{Y}^{+}(t_{n+1};\ \alpha) \approx \overline{Y}^{+}(t_n;\ \alpha) + h\,\mathrm{G}[\mathrm{t}_n, \underline{Y}^{+}(\mathrm{t}_n;\alpha), \overline{Y}^{+}(\mathrm{t}_n;\alpha)],$$

$$\underline{Y}^{-}(t_{n+1};\ \beta) \approx \underline{Y}^{-}(t_n;\ \beta) + hH[\mathrm{t}_n, \underline{Y}^{-}(\mathrm{t}_n;\beta), \overline{Y}^{-}(\mathrm{t}_n;\beta)],$$

$$\overline{Y}^{-}(t_{n+1};\ \beta) \approx \overline{Y}^{-}(t_n;\ \beta) + hI[\mathrm{t}_n, \underline{Y}^{-}(\mathrm{t}_n;\beta), \overline{Y}^{-}(\mathrm{t}_n;\beta)].$$

and

$$\underline{y}^{+}(t_{n+1};\ \alpha) = \underline{y}^{+}(t_n;\ \alpha) + h\,\mathrm{F}[\mathrm{t}_n, \underline{y}^{+}(\mathrm{t}_n;\alpha), \overline{y}^{+}(\mathrm{t}_n;\alpha)],$$

$$\overline{Y}^{+}(t_{n+1};\ \alpha) = \overline{y}^{+}(t_n;\ \alpha) + h\,\mathrm{G}[\mathrm{t}_n, \underline{y}^{+}(\mathrm{t}_n;\alpha), \overline{y}^{+}(\mathrm{t}_n;\alpha)],$$

$$\underline{Y}^{-}(t_{n+1};\ \beta) = \underline{y}^{-}(t_n;\ \beta) + hH[\mathrm{t}_n, \underline{y}^{-}(\mathrm{t}_n;\beta), \overline{y}^{-}(\mathrm{t}_n;\beta)],$$

$$\overline{Y}^{-}(t_{n+1};\ \beta) = \overline{y}^{-}(t_n;\ \beta) + hI[\mathrm{t}_n, \underline{y}^{-}(\mathrm{t}_n;\beta), \overline{y}^{-}(\mathrm{t}_n;\beta)].$$

We need the following lemmas to show the convergence of these approximates, that is, $\underline{y}^{+}(\mathrm{t}_n;\alpha)$, $\overline{y}^{+}(\mathrm{t}_n;\alpha), \underline{y}^{-}(\mathrm{t}_n;\alpha)$, and $\overline{y}^{-}(\mathrm{t}_n;\alpha)$ converges to $\underline{Y}^{+}(\mathrm{t}_n;\alpha)$, $\overline{Y}^{+}(\mathrm{t}_n;\alpha), \underline{Y}^{-}(\mathrm{t}_n;\alpha)$, and $\overline{Y}^{-}(\mathrm{t}_n;\alpha)$ respectively whenever $h \to 0$.

Lemma 5.1:

Let the sequence of numbers $\{W_n^{+}\}_{n=0}^{N}, \{W_n^{-}\}_{n=0}^{N}$ satisfy $|W_{n+1}| \le A|W_n| + B \quad 0 \le n \le N-1$ for some given positive constants A and B. Then $|W_n| \le A^n|W_0| + B\dfrac{A^n - 1}{A-1}$.

Lemma 5.2:

Let the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le |W_n| + A.\max\{|W_n||V_n|\} + B,$$

$$|V_{n+1}| \le |V_n| + A.\max\{|W_n||V_n|\} + B,$$

for some given positive constants A and B and denote $U_n = |W_n| + |V_n|, \quad 0 \le n \le N$.

Then $U_n \le \overline{\mathrm{A}}^{\mathrm{n}} U_0 + \overline{\mathrm{B}}\,\dfrac{\overline{\mathrm{A}}^{\mathrm{n}} \text{-}1}{\overline{\mathrm{A}}\text{-}1}, \quad 0 \le n \le N$ where $\overline{\mathrm{A}} = 1+2\mathrm{A}$ and $\overline{\mathrm{B}} = 2\mathrm{B}$..

Let F (t, u, v) and G (t, u, v) are obtained by substituting $[y(t)]_r = [u, v]$ in (5.2).

The domain where F and G are defined is therefore

$$K = \{(t,u,v)/0 \le t \le T, \quad -\infty < v < \infty,\ -\infty < u \le v\}.$$

Theorem 5.1:

Let F (t, u, v), G (t, u, v), H (t, u, v) and I(t, u, v) belong to $c^{P}(K)$ and let the partial derivatives of F and G be bounded over K. Then for arbitrary fixed $\alpha, \beta : 0 \le \alpha, \beta \le 1$, the approximate solution of (4.1), $\left[\underline{y}^{+}(\mathrm{t}_n;\alpha),\ \overline{y}^{+}(\mathrm{t}_n;\alpha), \underline{y}^{-}(\mathrm{t}_n;\beta),\ \overline{y}^{-}(\mathrm{t}_n;\beta)\right]$ converges to the exact solution $\left[\underline{Y}^{+}(\mathrm{t}_n;\alpha),\ \overline{Y}^{+}(\mathrm{t}_n;\alpha), \underline{Y}^{-}(\mathrm{t}_n;\beta),\ \overline{Y}^{-}(\mathrm{t}_n;\beta)\right]$.

Proof:

By using Taylor's theorem

$$\underline{Y}^{+}(t_{n+1};\alpha)=\underline{Y}^{+}(t_n;\alpha)+hF(t_n,\underline{Y}^{+}(t_n;\alpha),\overline{Y}^{+}(t_n;\alpha))+\frac{h^{p+1}c_i^{\,p}}{(p+1)!}\underline{Y}^{+(p+1)}(\xi_{n,1})$$

$$\overline{Y}^{+}(t_{n+1};\alpha)=\overline{Y}^{+}(t_n;\alpha)+hG(t_n,\underline{Y}^{+}(t_n;\alpha),\overline{Y}^{+}(t_n;\alpha))+\frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y}^{+(p+1)}(\xi_{n,2})$$

$$\underline{Y}^{-}(t_{n+1};\beta)=\underline{Y}^{-}(t_n;\beta)+hH(t_n,\underline{Y}^{-}(t_n;\beta),\overline{Y}^{-}(t_n;\beta))+\frac{h^{p+1}c_i^{\,p}}{(p+1)!}\underline{Y}^{-(p+1)}(\xi_{n,3})$$

$$\overline{Y}^{-}(t_{n+1};\beta)=\overline{Y}^{-}(t_n;\beta)+hI(t_n,\underline{Y}^{-}(t_n;\beta),\overline{Y}^{-}(t_n;\beta))+\frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y}^{-(p+1)}(\xi_{n,4})$$

Where $\xi_{n,1},\xi_{n,2},\xi_{n,3},\xi_{n,4}\in(t_n,t_{n+1})$

Now if we denote

$$W_n^{+}=\underline{Y}^{+}(t_n,\alpha)-\underline{y}^{+}(t_n,\alpha)\ ,\ \ V_n^{+}=\overline{Y}^{+}(t_n,\alpha)-\overline{y}^{+}(t_n,\alpha)$$

$$W_n^{-}=\underline{Y}^{-}(t_n,\beta)-\underline{y}^{-}(t_n,\beta)\ \text{ and }\ V_n^{-}=\overline{Y}^{-}(t_n,\beta)-\overline{y}^{-}(t_n,\beta)$$

Then the above two expressions converted to
Hence we can write

$$\left|W_{n+1}^{+}\right|\le\left|W_n^{+}\right|+2Lh\max\left(\left|W_n^{+}\right|,\left|V_n^{+}\right|\right)+\frac{h^{p+1}}{(p+1)!}\mathrm{M}_1$$

$$\left|V_{n+1}^{+}\right|\le\left|V_n^{+}\right|+2Lh\max\left(\left|W_n^{+}\right|,\left|V_n^{+}\right|\right)+\frac{h^{p+1}}{(p+1)!}\mathrm{M}_1$$

$$\left|W_{n+1}^{-}\right|\le\left|W_n^{-}\right|+2Lh\max\left(\left|W_n^{-}\right|,\left|V_n^{-}\right|\right)+\frac{h^{p+1}}{(p+1)!}\mathrm{M}_2$$

$$\left|V_{n+1}^{-}\right|\le\left|V_n^{-}\right|+2Lh\max\left(\left|W_n^{-}\right|,\left|V_n^{-}\right|\right)+\frac{h^{p+1}}{(p+1)!}\mathrm{M}_2$$

Where

$$M_1=\max\left\{\max\left|\underline{Y}^{+(p+1)}(t,\alpha)\right|,\max\left|\overline{Y}^{+(p+1)}(t,\alpha)\right|\right\}$$

$$M_2=\max\left\{\max\left|\underline{Y}^{-(p+1)}(t,\beta)\right|,\max\left|\overline{Y}^{-(p+1)}(t,\beta)\right|\right\}\quad for\, t\in[0,\mathrm{T}],$$

And L > 0 is a bound from the partial derivative of F and G.
Therefore we can write,

$$\left|W_n^{+}\right|\le(1+4\mathrm{Lh})^n\left|U_0^{+}\right|+\left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\mathrm{Lh})^n-1}{4Lh},$$

$$\left|V_n^{+}\right|\le(1+4\mathrm{Lh})^n\left|U_0^{+}\right|+\left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\mathrm{Lh})^n-1}{4Lh}$$

$$\left|W_n^{-}\right|\le(1+4\mathrm{Lh})^n\left|U_0^{-}\right|+\left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\mathrm{Lh})^n-1}{4Lh},$$

$$\left|V_n^{-}\right|\le(1+4\mathrm{Lh})^n\left|U_0^{-}\right|+\left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\mathrm{Lh})^n-1}{4Lh}$$

where $\left|U_0^{+}\right|=\left|W_0^{+}\right|+\left|V_0^{+}\right|$ and $\left|U_0^{-}\right|=\left|W_0^{-}\right|+\left|V_0^{-}\right|$

In particular

$$\left|W_n^+\right| \le (1+4\mathrm{Lh})^N \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh},$$

$$\left|V_n^+\right| \le (1+4\mathrm{Lh})^N \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}$$

$$\left|W_n^-\right| \le (1+4\mathrm{Lh})^N \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh},$$

$$\left|V_n^-\right| \le (1+4\mathrm{Lh})^N \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}$$

Where

$$M = \max\{M_1, M_2\} \quad for\, t \in [0,\mathrm{T}],$$

Since $W_0^+ = V_0^+ = W_0^- = V_0^- = 0,$ we have

$$\left|W_n^+\right| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p, \quad \left|V_n^+\right| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p,$$

$$\left|W_n^-\right| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p, \quad \left|V_n^-\right| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p$$

Thus if $h \to 0$ we get $W_n^+ \to 0,\ V_n^+ \to 0,\ W_n^- \to 0$ and $V_n^- \to 0$ which completes the proof.

6. Numerical Examples:

Example 6.1:

Consider the FODE with $y'(t) = y \; with \; y(0) = (3,5,7;1.5,5,8)$

Solution:

The solution is given by

$$\underline{y}^+(t,\alpha) = (3+2\alpha)e^t \quad \overline{y}^+(t,\alpha) = (7-2\alpha)e^t$$

$$\underline{y}^-(t,\beta) = (5-3.5\beta)e^t \quad \overline{y}^-(t,\beta) = (5+3\beta)e^t$$

The numerical solution and Error results for example 6.1 at t =1 and $(\alpha,\beta)=1$ are shown in the Table 6.1 and Table 6.2 The solution graphs are shown Figure 6.1 andFigure 6.2.

Table 6.1

α,β	**Absolute Error for IFRK3 at t=1**			
	$\underline{y}^+(t;\alpha)$	$\overline{y}^+(t;\alpha)$	$\underline{y}^-(t;\beta)$	$\overline{y}^-(t;\beta)$
0.00	3.633e-02	8.477e-02	1.817e-02	9.688e-02
0.20	4.117e-02	7.993e-02	2.664e-02	8.961e-02
0.40	4.602e-02	7.508e-02	3.512e-02	8.235e-02
0.60	5.086e-02	7.024e-02	4.360e-02	7.508e-02
0.80	5.571e-02	6.539e-02	5.207e-02	6.782e-02
1.00	6.055e-02	6.055e-02	6.055e-02	6.055e-02

Table 6.2

α,β	**Absolute Error for IFRK4 at t=1**			
	$\underline{y}^+(t;\alpha)$	$\overline{y}^+(t;\alpha)$	$\underline{y}^-(t;\beta)$	$\overline{y}^-(t;\beta)$
0.00	5.414e-08	1.263e-07	2.707e-08	1.444e-07
0.20	6.136e-08	1.191e-07	3.971e-08	1.336e-07
0.40	6.858e-08	1.119e-07	5.234e-08	1.227e-07
0.60	7.580e-08	1.047e-07	6.497e-08	1.119e-07
0.80	8.302e-08	9.746e-08	7.761e-08	1.011e-07
1.00	9.024e-08	9.024e-08	9.024e-08	9.024e-08

Figure 6.1

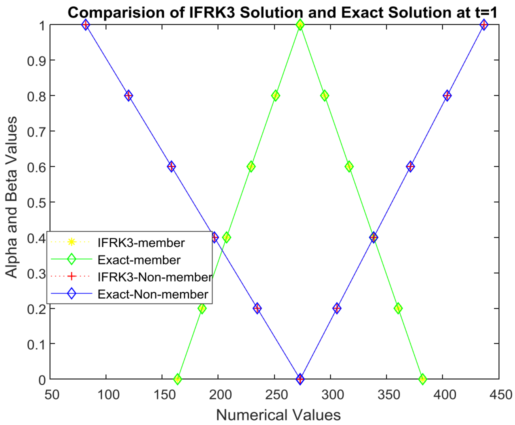
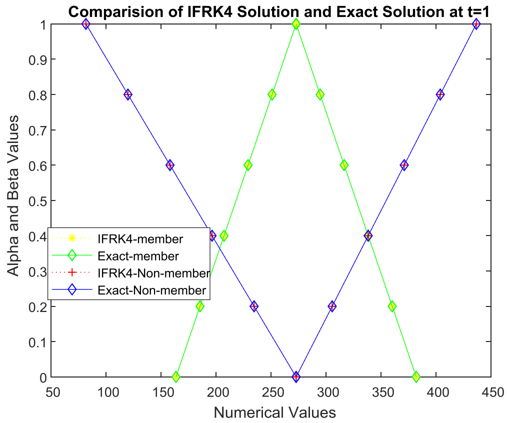


Figure 6.2



Conclusion:

In this paper, the Runge-Kutta methods of order three and four have been proposed to solve the intuitionistic fuzzy IVPs. The convergence analysis of these methods has been discussed. The applicability of these methods has been illustrated through examples of intuitionistic fuzzy IVPs. From the numerical results, it is observed that the absolute error is negligibly small and the accuracy increases as the order of the RK method increases. Hence, it is seen that the RK method is suitable for solving intuitionistic fuzzy IVPs.

References:

1. S. Abbasbandy, T. Allah Viranloo, Numerical solutions of fuzzy differential equations by Taylor method, Computational Methods in Applied Mathematics, 2 (2002) 113-124.
2. S. Abbasbandy, T. Allah Viranloo, Numerical solution of fuzzy differential equation by Runge Kutta method and intuitionistic treatment, Notes on IFS, 8(3) (2002) 45-53.
3. A. Z. Ahmad, M. K. Hasan, A new fuzzy version of Euler’s method for solving differential equations with fuzzy initial values,SainsMalasiana,40(6) (2011) 651-657.
4. T. Allahviranloo, S. Abbasbandy, N. Ahmady, E. Ahmady, Improved predictor-corrector method for solving fuzzy initial value problems, Information Sciences,179 (2009) 945-955.
5. K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems, 20 (1986) 87-96.
6. B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy-number valued functions with applications to fuzzy differential equations, Fuzzy sets and systems 151 (2005) 581- 599.
7. B. Bede, I. J. Rudas, A. L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Information Sciences,177 (2007), 1648-1662.
8. S. L. Chang and L. A. Zadeh, On Fuzzy Mapping and Control, IEEE Trans. System Man Cybernet, 2, (1972) 30-34.
9. O. Kaleva, Fuzzy differential equations, Fuzzy sets and systems, 24 (1987), 301-317.
10. O. Kaleva, The Cauchy problem for Fuzzy Differential Equations, Fuzzy Sets Systems, 35(1990), 389-396.
11. S. Melliani, L. S. Chadli, Intuitionistic fuzzy differential equation, Notes on IFS, 6 (2) (2000), 37-41.
12. M. Ma, M. Friedman, A. Kandel, Numerical solutions of fuzzy differential equations, Fuzzy sets and Systems, 105 (1999), 133-138.
13. V. Nirmala, S. Chenthur Pandian, Numerical Solution of Fuzzy Differential Equation by Fourth Order Runge-Kutta Method with Higher Order Derivative Approximations, European Journal of Scientific Research, 62(2) (2011) 198-206.
14. V. Nirmala, S. ChenthurPandian, Numerical Approach for Solving Intuitionistic Fuzzy Differential Equation Under Generalised Differentiability concept, Applied Mathematical Sciences,9(67) (2015), 3337-3346.
15. V. Nirmala, V. Parimala, P. Rajarajeswari, Modified Euler Method for Finding Numerical Solution of Intuitionistic Fuzzy Differential Equation under Generalised Differentiability Concept, International Journal of Applied Engineering Research, 10(72) (2015), 40-45.
16. V. Nirmala, V. Parimala, P. Rajarajeswari, Intuitionistic Fuzzy Differential Equation and Numerical Solution by Fourth order Runge-Kutta Method under Generalized Differentiability, International Journal of Applied Engineering Research,10(72) (2015), 46-51.
17. S. Seikkala, On the Fuzzy Initial Value Problem, Fuzzy Sets Systems, 24 (1987), 319-330.
18. Sneh Lata, Amit Kumar, A new method to solve time dependent intuitionistic fuzzy differential equations and its application to analyze the intuitionistic fuzzy reliability of industrial systems, SAGA, Concurrent engineering published online 23 July (2012).

19. Sankar Prasad Mondal, Tapan Kumar Roy, First order homogeneous ordinary differential equation with initial value as triangular intuitionistic fuzzy number, Journal of Uncertainty in Mathematics Science, (2014), 1-17. DOI:10.5899/2014/jums-00003.
20. L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965), 338-353.